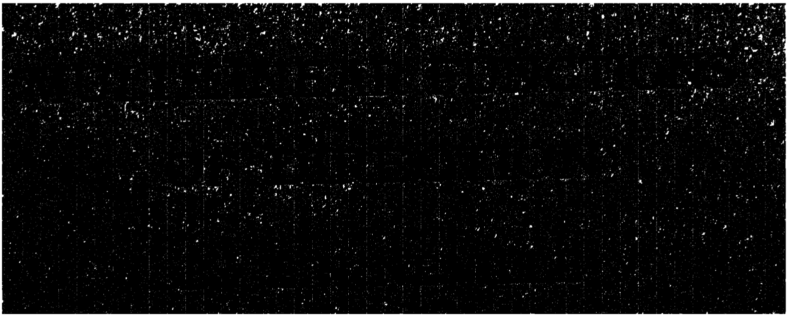

MULTILINEAR FUNCTIONS OF DIRECTION

CAMBRIDGE UNIVERSITY PRESS
C. F. CLAY, MANAGER
LONDON : FETTER LANE, E.C. 4

NEW YORK : THE MACMILLAN CO.
BOMBAY, CALCUTTA, MADRAS: MACMILLAN AND CO., LTD.
TORONTO : THE MACMILLAN CO. OF CANADA, LTD.
TOKYO: MARUZEN-KABUSHIKI-KAISHA

MULTILINEAR FUNCTIONS OF DIRECTION

AND THEIR USES IN DIFFERENTIAL GEOMETRY

BY

ERIC HAROLD NEVILLE

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE
PROFESSOR OF MATHEMATICS IN UNIVERSITY COLLEGE, READING

CAMBRIDGE
AT THE UNIVERSITY PRESS
1921

PREFACE

THE distinctive feature of this work is that the functions discussed are primarily not functions of a single variable direction but functions of several independent directions. Functions of a single direction emerge when the directions originally independent become related, and a large number of elementary theorems of differential geometry express in different terms a few properties of a few simple functions; since one of the objects of the essay is to emphasise the coordinating power of the theory, the presence of many results with which every reader will be thoroughly familiar calls for no apology.

In the applications to the geometry of a single surface two functions thought to be new are described. The first, studied in Section 4, depends on two tangential directions, reduces to normal curvature when these directions coincide, and is called here bilinear curvature. I became acquainted with this function in 1911 and used it in lectures early in 1914. The second, the subject of Section 6, depends on three directions, and reduces to the cubic function associated with the name of Laguerre; the function is symmetrical, and because the equations of Codazzi can be read as asserting its symmetry I have called the general function the Codazzi function.

The theory of multilinear functions does not merely coordinate. It affords simple proofs of the relations between the cubic functions of Laguerre and Darboux (6·231, 6·234) and of formulae (7·242, 7·351, 7·352) for the twist of a family of surfaces, and it leads naturally to expressions (7·241) for the rates of change of the two principal curvatures of a variable member of a family of surfaces at the current point of an orthogonal trajectory of the family, expressions that are interesting because their existence was deduced by Forsyth in 1903 from an enumeration of invariants.

E. H. N.

June, 1920.

NOTE

For the sake of brevity, the space considered is real, but the restriction operates only to the same extent as in other branches of differential geometry. If it is removed, the intrinsic distinction between the positive square root and the negative square root of a given uniform function has to be replaced by a more artificial distinction based on a dissection that is to some extent arbitrary. And there is always a possibility that results need modification if isotropic lines or planes are involved; as a rule, nul *vectors* are admissible as arguments but nul *directions* are not.

CONTENTS

MULTILINEAR FUNCTIONS OF DIRECTION

Preliminary Paragraphs

0·1. The association of a direction OR with a real number r, which may be positive, zero, or negative, determines a vector which will be denoted by r_R; of this vector r will be called the *amount* in the direction OR. The vector r_R possesses in addition to the direction OR the reverse direction, which we shall denote consistently by OR', and the amount of r_R in the direction OR' is $-r$. The zero vector has all directions, and its amount in every direction is zero; a proper vector has only two directions and two amounts.

A vector of amount unity is called a unit vector or *radial.* The vector 1_R has the direction OR' as well as the direction OR, but there is no confusion in describing the direction OR as *the* direction of the radial.

0·2. There is an infinity of angles between two directions in space, but these angles all have the same cosine. If ϵ_{RS} is an angle between directions OR, OS of two vectors $\mathbf{r}$, $\mathbf{s}$ whose amounts in these directions are r, s, the product $rs \cos \epsilon_{RS}$ depends only on $\mathbf{r}$ and $\mathbf{s}$, not on any choice which is arbitrary when the vectors are given; this product will be called the *projected product** of $\mathbf{r}$ and $\mathbf{s}$ and denoted by $\mathscr{S}\mathbf{rs}$. The projected product of a vector $\mathbf{s}$ and a radial 1_R is the projection of $\mathbf{s}$ in the direction OR, and the projected product of two radials is the cosine of the angles between their directions.

0·3. Any three vectors $\mathbf{p}^1$, $\mathbf{p}^2$, $\mathbf{p}^3$ which are not coplanar form a vector frame, in which the arbitrary vector $\mathbf{r}$ is determined by the three scalars ξ, η, ζ such that

$$\mathbf{r} = \xi\mathbf{p}^1 + \eta\mathbf{p}^2 + \zeta\mathbf{p}^3.$$

* Many writers have not hesitated to call this the scalar product, although the function is the negative of that for which Hamilton designed the name. There is no universal notation; to transfer the letter as well as the name rendered familiar by Hamilton and to appropriate brackets of some special kind are courses equally open to criticism, and if there is here a vacant rôle in the symbolism of vector analysis it is one for which the initial of Gibbs and Grassmann may be cast with peculiar fitness. Neither $\mathbf{r}\,.\,\mathbf{s}$ nor $\mathbf{r}\times\mathbf{s}$ is quite secure from misunderstanding, since Heaviside uses the one for a dyad and Gibbs the other for a vector product; I am conservative enough to regard $\mathbf{rs}$ as denoting a quaternion.

The polar of the frame $\mathbf{p}^1\mathbf{p}^2\mathbf{p}^3$ is the frame $\bar{\mathbf{p}}^1\bar{\mathbf{p}}^2\bar{\mathbf{p}}^3$ such that $\mathscr{S}\bar{\mathbf{p}}^h\bar{\mathbf{p}}^k$ is unity or zero according as h and k are the same or different, that is to say, such that $\bar{\mathbf{p}}^1$ is at right angles to both $\mathbf{p}^2$ and $\mathbf{p}^3$ and $\mathscr{S}\mathbf{p}^1\bar{\mathbf{p}}^1$ is unity, $\bar{\mathbf{p}}^2$ is at right angles to both $\bar{\mathbf{p}}^3$ and $\bar{\mathbf{p}}^1$ and $\mathscr{S}\bar{\mathbf{p}}^2\mathbf{p}^2$ is unity, and $\bar{\mathbf{p}}^3$ is at right angles to both $\bar{\mathbf{p}}^1$ and $\bar{\mathbf{p}}^2$ and $\mathscr{S}\mathbf{p}^3\bar{\mathbf{p}}^3$ is unity. If

$$\mathbf{r} = \lambda\bar{\mathbf{p}}^1 + \mu\bar{\mathbf{p}}^2 + \nu\bar{\mathbf{p}}^3,$$

then
$$\mathscr{S}\mathbf{r}\mathbf{p}^1 = \lambda, \quad \mathscr{S}\mathbf{r}\mathbf{p}^2 = \mu, \quad \mathscr{S}\mathbf{r}\mathbf{p}^3 = \nu,$$

and since the relation between the two vector frames is reciprocal,

$$\mathscr{S}\mathbf{r}\bar{\mathbf{p}}^1 = \xi, \quad \mathscr{S}\mathbf{r}\bar{\mathbf{p}}^2 = \eta, \quad \mathscr{S}\mathbf{r}\bar{\mathbf{p}}^3 = \zeta;$$

considered as derived from the frame $\mathbf{p}^1\mathbf{p}^2\mathbf{p}^3$, the projected products λ, μ, ν are naturally called the polar coefficients of $\mathbf{r}$.

0·4. When we have occasion to use a Cartesian frame of reference, we shall not assume it to be trirectangular. We shall use α, β, γ for angles between the axes of reference and A, B, Γ for angles between the planes, A being an angle from the second plane to the third round the first axis just as α is an angle from the second axis to the third in the first plane; A, B, Γ are *external* angles of the spherical triangle of which α, β, γ are sides. Also we shall denote by Υ the sine of this triangle, that is, we shall write

$$\Upsilon = \sin\beta\sin\gamma\sin \mathrm{A} = \sin\gamma\sin\alpha\sin \mathrm{B} = \sin\alpha\sin\beta\sin\Gamma.$$

Then if x, y, z are the components and l, m, n the projections of any vector,

0·41
$$\left[\begin{aligned} l &= x + y\cos\gamma + z\cos\beta, \\ m &= x\cos\gamma + y + z\cos\alpha, \\ n &= x\cos\beta + y\cos\alpha + z, \end{aligned}\right.$$

and on the other hand

0·42
$$\left[\begin{aligned} x &= l\Upsilon^{-2}\sin^2\alpha + m\Upsilon^{-1}\cot\Gamma + n\Upsilon^{-1}\cot \mathrm{B}, \\ y &= l\Upsilon^{-1}\cot\Gamma + m\Upsilon^{-2}\sin^2\beta + n\Upsilon^{-1}\cot \mathrm{A}, \\ z &= l\Upsilon^{-1}\cot \mathrm{B} + m\Upsilon^{-1}\cot \mathrm{A} + n\Upsilon^{-2}\sin^2\gamma. \end{aligned}\right.$$

The projected square of the vector, having the value $lx + my + nz$, can be expressed as a quadratic function of components alone by means of 0·41 or of projections alone by means of 0·42; thus

0·43 $$r^2 = x^2 + y^2 + z^2 + 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma,$$

but in terms of projections alone r^2 is most readily given by means of a determinant; eliminating x, y, z between 0·41 and

$$r^2 = lx + my + nz$$

we have

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & l \\ \cos\gamma & 1 & \cos\alpha & m \\ \cos\beta & \cos\alpha & 1 & n \\ l & m & n & r^2 \end{vmatrix} = 0,$$

that is

0·44 $$r^2 = -\Upsilon^{-2} \begin{vmatrix} 1 & \cos\gamma & \cos\beta & l \\ \cos\gamma & 1 & \cos\alpha & m \\ \cos\beta & \cos\alpha & 1 & n \\ l & m & n & 0 \end{vmatrix}.$$

If OP is a direction perpendicular to each of two directions OR, OS and if ϵ_{RS} is an angle from OR to OS round OP, the *cosines* of OP are given in terms of the *ratios* of OR and OS by

0·45 $$(l_P,\ m_P,\ n_P) \sin\epsilon_{RS} = \Upsilon \begin{Vmatrix} x_R & y_R & z_R \\ x_S & y_S & z_S \end{Vmatrix}$$

and the *ratios* of OP in terms of the *cosines* of OR and OS by

0·46 $$(x_P,\ y_P,\ z_P) \sin\epsilon_{RS} = \Upsilon^{-1} \begin{Vmatrix} l_R & m_R & n_R \\ l_S & m_S & n_S \end{Vmatrix}.$$

The components x, y, z, and the projections l, m, n, of a vector $\mathbf{r}$ in the Cartesian frame $OABC$ are the coefficients and the polar coefficients of $\mathbf{r}$ in the vector frame composed of the radials 1_A, 1_B, 1_C. But it must be observed that the polar of this vector frame is not as a rule the Cartesian frame polar to $OABC$ but consists of vectors of amounts $\Upsilon^{-1}\sin\alpha$, $\Upsilon^{-1}\sin\beta$, $\Upsilon^{-1}\sin\gamma$.

0·5. For the comparison of directions in one plane actual angles can be used, a definite direction of angular measurement being adopted. An angle from OS to OT will be denoted by ϵ_{ST}; this angle is not free from ambiguity, for any restriction on the magnitude or sign of angles is not merely superfluous but irksome, but $\cos\epsilon_{ST}$ and $\sin\epsilon_{ST}$ are determinate functions of the two directions OS, OT, and so also is the rate of change of ϵ_{ST} with respect to any variable on which the directions depend in a regular manner.

When axes of reference $A'OA$, $B'OB$ are being used in the plane, an angle ϵ_{AB} will be denoted by ω. To deal simply and symmetrically with a variable direction OT, angles ϵ_{AT}, ϵ_{TB} are both required; the sum $\epsilon_{AT}+\epsilon_{TB}$ must differ from ω by an integral multiple of 2π, and α, β, or if necessary α_T, β_T, will be used for a pair of angles ϵ_{AT}, ϵ_{TB} subject to the convention $\alpha+\beta=\omega$.

The theory of multilinear functions of direction in a plane persistently associates with each direction one of the perpendicular directions, and the direction which makes a positive right angle with OT will be denoted by OE or by OE_T; for OE_S will be substituted OD.

0·6. A function $F(T)$ of the direction OT in a plane regarded as a function $F(\epsilon_{WT})$ of an angle to OT from a fixed direction OW, requires for its study its derivative $dF(\epsilon_{WT})/d\epsilon_{WT}$. This derivative is itself a function of ϵ_{WT}, that is, of the direction OT, but since it does not really depend on OW it may be called simply the angular derivative of $F(T)$ and will be denoted by $daF(T)$:

0·61 $$daF(T)=\lim_{S\to T}[\{F(S)-F(T)\}/\epsilon_{TS}].$$

A function of a number of independent directions in a plane has an angular derivative with respect to each of them, and the various angular derivatives of $F(Q, R, \ldots)$ will be written $da_Q F(Q, R, \ldots)$, $da_R F(Q, R, \ldots)$, and so on.

If the directions OQ, OR, ... in a plane are made dependent on a direction OT in that plane, the function $F(Q, R, \ldots)$ becomes a function of OT, having an angular derivative with respect to OT given by
$$\frac{\partial F}{\partial\epsilon_{WQ}}\frac{d\epsilon_{WQ}}{d\epsilon_{WT}}+\frac{\partial F}{\partial\epsilon_{WR}}\frac{d\epsilon_{WR}}{d\epsilon_{WT}}+\ldots.$$
The dependence of OQ, OR, ... on OT is a dependence of angles ϵ_{TQ}, ϵ_{TR}, ... on OT, and since
$$\epsilon_{WQ}=\epsilon_{WT}+\epsilon_{TQ},\ \epsilon_{WR}=\epsilon_{WT}+\epsilon_{TR},\ \ldots,$$
the derivatives $d\epsilon_{WQ}/d\epsilon_{WT}$, $d\epsilon_{WR}/d\epsilon_{WT}$, ... have the values
$$1+da\,\epsilon_{TQ},\ 1+da\,\epsilon_{TR},\ \ldots,$$
and

0·62. *The angular derivative with respect to OT of a function $F(Q, R, \ldots)$ of directions themselves dependent on OT is*
$$(1+da\,\epsilon_{TQ})\,da_Q F+(1+da\,\epsilon_{TR})\,da_R F+\ldots.$$

In particular

0·63. *If the directions OQ, OR, ... make constant angles with OT, the angular derivative of a function $F(Q, R, \ldots)$ with respect to OT is the sum of the several angular derivatives $da_Q F$, $da_R F$, ...*

An angular derivative in space is a function of two directions at right angles. If ON is a direction at right angles to OT, the direction of angular measurement in the plane to which ON is normal is related to ON by the spatial convention; a function of direction in space becomes by the restriction of its argument to the plane normal to ON a function of direction in that plane, with an angular derivative whose value at OT depends no less on ON than on OT.

1. Linear and Multilinear Functions

1·11. Intrinsically, a linear function of a variable vector is a function whose value for the sum of two vectors, and therefore also for the sum of any finite number of vectors, is the sum of its values for the several components; a multilinear function is a function of a number of independent vectors that is linear in each of them.

1·12. The value of any linear function for the argument r_R is r times the value of the same function for the argument 1_R. If a frame of reference $OABC$ is used and the components of the variable vector $\mathbf{r}$ are x, y, z, then since $\mathbf{r}$ is $x1_A + y1_B + z1_C$, a function $F(\mathbf{r})$ which is linear is necessarily expressible as

$$xF(1_A) + yF(1_B) + zF(1_C),$$

and conversely a function of $\mathbf{r}$ which is of the form $xL + yM + zN$ where L, M, N do not depend on $\mathbf{r}$ must be linear:

1·121. *A linear function of the vector* $\mathbf{r}$ *is a function which is a homogeneous linear function of the components of* $\mathbf{r}$ *in any frame.*

1·13. A function whose arguments are radials may be regarded as a function simply of direction, and a function of direction is said to be linear if the function of r and OR obtained by multiplying its value for the direction OR by the number r is a linear function of the vector r_R. We can if we wish avoid the explicit mention of vectors in the definition of a linear function of direction, either by introducing implicitly the definition of the sum of two vectors or

by using a frame of reference. A function of direction is linear if given any two successive steps OQ, QR, of lengths p, q, whose resultant OR has length r, the sum of p times the value of the function for the direction OQ and q times the value of the function for the direction QR is r times the value of the function for the direction OR. And a function of direction is linear if it is expressible as a homogeneous linear function of the ratios of the direction with reference to any frame; with this last definition we have to notice that a function of the direction ratios which is not *given* as a homogeneous linear function may in fact be expressible in this form in virtue of the quadratic identity to which the ratios are necessarily subject.

1·14. The definitions of linear and multilinear functions of vectors and directions are designed to restrict as little as possible the nature of the function. In the present work the ultimate equations are scalar, but vectors and other functions are essential to the processes.

1·21. For an arbitrary function of the k vectors $\mathbf{r}_1, \mathbf{r}_2, \dots \mathbf{r}_k$, the natural notation is of the form $F(\mathbf{r}_1, \mathbf{r}_2, \dots \mathbf{r}_k)$, but for a function that is multilinear there is more even than brevity to be gained by substituting the form $P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k$, or $P^k\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k$ if the degree of the function has to be made prominent, for this form emphasises the identities such as

$$P(\mathbf{s}+\mathbf{t})\,\mathbf{r}_2 \dots \mathbf{r}_k = P\,\mathbf{s}\mathbf{r}_2 \dots \mathbf{r}_k + P\,\mathbf{t}\mathbf{r}_2 \dots \mathbf{r}_k$$

involved in the definitions. It must be remembered that the order in which the vectors are written is not irrelevant unless the function is symmetrical in its definition: a function defined unsymmetrically may be in fact symmetrical, in which event the order of writing the variables does not affect the truth of any formulae but the assertion of the symmetry is in itself significant.

1·22. The function of direction $P\mathbf{1}_A\mathbf{1}_B \dots \mathbf{1}_K$ is often denoted by $P_{AB\,.K}$. Were an attempt made to deal with functions of direction without mention of vectors this compact alternative would be used throughout, but since as a rule the sum of two radials and the rate of change of a variable radial are vectors that are not radials, the operations that are most natural commonly involve functions of directions and functions of vectors in one equation, and the effect

of too persistent a substitution of $P_{AB\ \ K}$ for $P1_A 1_B \dots 1_K$ is unsightly.

1·31. The advantages of detaching the symbol P from the group $P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k$ are secured by the method of Russell: P denotes the *relation* of a value of the function to the set of vectors on which the value depends, and is called a multilinear relation. The relation P will be described as the *core* of the function $P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k$ and of the corresponding function of direction $P_{AB\ \ K}$.

1·32. To prove that

1·321. *The sum of any finite number of functions multilinear in the same set of vectors is itself a multilinear function of the set*

and that

1·322. *The product of a multilinear function by any scalar is a multilinear function*

is easy. These propositions give further justification of the notation we are using, and provide a basis for definitions of the addition of cores and of the multiplication of a core by a scalar:

1·323 $$(\Sigma P)\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k = \Sigma\,(P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k),$$

1·324 $$(rP)\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k = r\,(P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_k).$$

1·33. In a multilinear function of degree k any h of the variable vectors or directions may play a parametric part. The function is then regarded as multilinear in the remaining $k-h$ variables, with a core which is a function of the h parameters, and we have only to compare 1·323 with the original definition of a multilinear function to see that this core is multilinear in the parameters; it is a multilinear function which is neither scalar nor vector. Thus the bilinear function $P\,\mathbf{rs}$ or P_{RS} yields two linear functions which are written as $(P{*}\mathbf{s})\,\mathbf{r}$ and $(P\mathbf{r}{*})\,\mathbf{s}$ or $(P_{*S})_R$ and $(P_{R*})_S$; if the degrees of the different functions are to be exhibited, the two linear functions of direction subsidiary to $P^2{}_{RS}$ are shewn as $(P^1{}_{*S})^1{}_R$ and $(P^1{}_{R*})^1{}_S$.

1·41. If $\chi_h{}^1, \chi_h{}^2, \chi_h{}^3$ are the coefficients of $\mathbf{r}_h$ in a vector frame $\mathbf{p}^1\mathbf{p}^2\mathbf{p}^3$, the linearity of a function $P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_{k-1}\mathbf{r}_k$ in $\mathbf{r}_k$ implies the equality

1·411 $$P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_{k-1}\mathbf{r}_k =$$
$$\chi_k{}^1 P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_{k-1}\mathbf{p}^1 + \chi_k{}^2 P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_{k-1}\mathbf{p}^2 + \chi_k{}^3 P\,\mathbf{r}_1\mathbf{r}_2 \dots \mathbf{r}_{k-1}\mathbf{p}^3,$$

and the expansion of each of the variable vectors in turn in the same way gives the result

1·412 $P\,\mathbf{r}_1\mathbf{r}_2 \ldots \mathbf{r}_{k-1}\mathbf{r}_k =$

$$\Sigma \chi_1^{m_1}\chi_2^{m_2}\ldots\chi_{k-1}^{m_{k-1}}\chi_k^{m_k}\,P\,\mathbf{p}^{m_1}\mathbf{p}^{m_2}\ldots\mathbf{p}^{m_{k-1}}\mathbf{p}^{m_k},$$

where each of the affixes $m_1, m_2, \ldots m_{k-1}, m_k$ stands for one of the three symbols 1, 2, 3, and the summation extends to the 3^k possible terms. To obtain a formula in terms of the projected products $\mathscr{G}\mathbf{r}_h\mathbf{p}^1$, $\mathscr{G}\mathbf{r}_h\mathbf{p}^2$, $\mathscr{G}\mathbf{r}_h\mathbf{p}^3$, all that is necessary is to remember that these projected products are the coefficients $\bar{\chi}_h^1, \bar{\chi}_h^2, \bar{\chi}_h^3$ of $\mathbf{r}_h$ in the frame $\bar{\mathbf{p}}^1\bar{\mathbf{p}}^2\bar{\mathbf{p}}^3$ polar to $\mathbf{p}^1\mathbf{p}^2\mathbf{p}^3$, whence

1·413 $P\,\mathbf{r}_1\mathbf{r}_2 \ldots \mathbf{r}_{k-1}\mathbf{r}_k =$

$$\Sigma\,(\mathscr{G}\mathbf{r}_1\mathbf{p}^{m_1})(\mathscr{G}\mathbf{r}_2\mathbf{p}^{m_2})\ldots(\mathscr{G}\mathbf{r}_{k-1}\mathbf{p}^{m_{k-1}})(\mathscr{G}\mathbf{r}_k\mathbf{p}^{m_k})\,P\,\bar{\mathbf{p}}^{m_1}\bar{\mathbf{p}}^{m_2}\ldots\bar{\mathbf{p}}^{m_{k-1}}\bar{\mathbf{p}}^{m_k}.$$

Particular cases of 1·412 and 1·413 are expressions for the multilinear function referred to a Cartesian frame, namely

1·421 $P\,\mathbf{r}_1\mathbf{r}_2 \ldots \mathbf{r}_k = \Sigma c_1^{m_1}c_2^{m_2}\ldots c_k^{m_k}P_{M_1M_2\ \ M_k},$

where c_h^1, c_h^2, c_h^3 are the components of $\mathbf{r}_h$ and M_h stands for A, B, or C according as m_h stands for 1, 2, or 3, and

1·422 $P\,\mathbf{r}_1\mathbf{r}_2 \ldots \mathbf{r}_k = \Sigma p_1^{m_1}p_2^{m_2}\ldots p_k^{m_k}P\,\mathbf{k}^{m_1}\mathbf{k}^{m_2}\ldots\mathbf{k}^{m_3},$

where p_h^1, p_h^2, p_h^3 are the projections of $\mathbf{r}_h$ and $\mathbf{k}^1, \mathbf{k}^2, \mathbf{k}^3$ are those vectors normal to the planes OBC, OCA, OAB whose projections on OA, OB, OC are unity.

1·43. From 1·412 follow two fundamental theorems:

1·431. *The value of a multilinear function is known for every set of vectors in space if it is given for every selection from any three vectors that are not coplanar;*

1·432. *A function that is multilinear is wholly symmetrical if it is symmetrical with respect to any three vectors or any three directions that are not coplanar.*

Because of the second of these results, any two groups of theorems which express the complete symmetry of the same multilinear function with respect to different sets of vectors may be regarded as equivalent: this is one of the ways in which results diverse in form are coordinated by the theory developed here.

The two theorems of the last paragraph assume the functions involved to be defined for all sets of vectors or directions in space.

If only a single plane is in question, it is sufficient in 1·431 for the selection to be made from two vectors in that plane but not collinear and in 1·432 for the symmetry to be established for two such vectors.

1·51. That

1·511. *The projected product of two vectors is a bilinear function of these vectors,*
and that

1·512. *The projection of a constant vector on a variable direction is a linear function of the direction,*
follow from the elementary distributive property of the projected product. The converse of these theorems is also true, for if a, b, c are scalars, $ax + by + cz$ is the projected product of the vector of components x, y, z and the vector of projections a, b, c, and if the former vector is the radial $\mathbf{1}_T$, the same sum represents the projection of the latter vector on OT:

1·513. *Every linear scalar function of the variable vector* **r** *can be exhibited in one way only as the projected product of* **r** *and a vector independent of* **r**, *and every linear scalar function of the variable direction OT in one way only as the projection on OT of a vector independent of OT.*
It is convenient in both cases to call the vector the *source* of the linear function.

1·52. The projected product of the sources of two linear functions affords the simplest example of a scalar which depends only on two cores, and if the cores are Q and R this projected product will be denoted simply by QR. If Q and R themselves involve variables QR is of course a function of these variables. Thus if from two bilinear functions Q_{AB}, R_{CD} are formed two linear functions $(Q_{A*})_B$ $(R_{*D})_C$, the projected product $Q_{A*} R_{*D}$ is a function of the directions OA, OD; it is in fact a bilinear function, and so can be used to form on the same principle an infinity of other functions, such for example as $Q_{\dagger B} Q_{\dagger *} R_{*D}$.

1·53. Any two cores of the same degree give rise to a function corresponding to the projected product of the sources of two linear functions, but in the absence of a direct definition of this function in general, we must describe the function defined in the

last paragraph in such a way as to indicate the line of extension. Referred to a frame $OABC$, the linear function $P\,\mathbf{r}$ can be expressed in the two forms

$$l^P x_{\mathbf{r}} + m^P y_{\mathbf{r}} + n^P z_{\mathbf{r}}, \quad x^P l_{\mathbf{r}} + y^P m_{\mathbf{r}} + z^P n_{\mathbf{r}},$$

where l^P, m^P, n^P are the projections and x^P, y^P, z^P the components of the source, and since the value of the projected product of two linear cores Q, R is given by the two sums

$$l^Q x^R + m^Q y^R + n^Q z^R, \quad x^Q l^R + y^Q m^R + z^Q n^R,$$

not only are these sums equal but their value is independent of the particular frame $OABC$. Similarly the multilinear function $P\,\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ whatever its degree can be expanded with reference to the frame $OABC$ in the two forms

$$\Sigma\, p^P{}_{m_1 m_2 \ldots m_k}\, c_1{}^{m_1} c_2{}^{m_2} \ldots c_k{}^{m_k}, \quad \Sigma\, c^P{}_{m_1 m_2 \ldots m_k}\, p_1{}^{m_1} p_2{}^{m_2} \ldots p_k{}^{m_k},$$

and if Q, R are any two cores of the same degree k the sums

$$\Sigma\, p^Q{}_{m_1 m_2 \ldots m_k}\, c^R{}_{m_1 m_2 \ldots m_k}, \quad \Sigma\, c^Q{}_{m_1 m_2 \ldots m_k}\, p^R{}_{m_1 m_2 \ldots m_k}$$

are equal and have a value independent of the frame $OABC$; this common value defines the *projected product of the cores* Q, R, and is denoted by QR.

1·54. When once the projected product QR is defined for cores of arbitrary degree, a whole group of functions is seen to be derivable from any two or more cores, or indeed from any one core of degree not less than two. For example, from a trilinear core P by regarding one direction OB as parametric we derive a bilinear core $P_{*B\dagger}$, and if Q is a bilinear core, the projected product $P_{*B\dagger}Q$, better denoted by $P_{*B\dagger}Q_{*\dagger}$, is itself a linear function of OB and gives rise by combination with any linear scalar core R to a projected product $(P_{*\S\dagger}\,Q_{*\dagger})\,R_{\S}$; without attempting to classify functions of this kind we must recognise their nature when they present themselves.

Symmetry reduces the number of distinct functions to which a given multilinear function is related. For example, if Q_{AB}, R_{CD} are unsymmetrical bilinear functions, the four bilinear functions $Q_{*S}R_{*T}$, $Q_{S*}R_{*T}$, $Q_{*S}R_{T*}$, $Q_{S*}R_{T*}$ are distinct, but if the original functions are both symmetrical, the four derived functions coincide.

1·55. If two multilinear functions $Q\,\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$, $R\,\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ of the same degree are defined only for vectors in a particular plane, the

projected product QR can be defined as in 1·53, with the sole difference that the frame of reference is two-dimensional. Moreover, if one multilinear function $Q\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ is defined only for vectors in a particular plane and another $R\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ of the same degree is defined without restriction on $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_k$, a projected product is definable by the restriction of the arguments of $R\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ to the plane in which they can serve as arguments to $Q\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ also, and no confusion can be caused by denoting this projected product by QR; the only point to be remembered is that if by a change in the definitions the restriction on the arguments of $Q\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ is subsequently removed, QR will be in danger of acquiring a second meaning inconsistent with the first.

1·61. Should the function $P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ involve any variables other than the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_k$, then if a change in these additional variables is not *necessarily* accompanied by a change in the vectors it is the core that is to be regarded as a function, and a limit of $P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ for variations in which $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_k$ are constant is a function of $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_k$ which if multilinear can be used to define a limit of P. It is difficult to be more precise in this assertion without placing undue restriction on its scope; the case which is for us important affords the best commentary; if $P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$ is a scalar or a vector depending on a scalar variable t in such a way that for each particular set of values of $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_k$ there is a derivative $d(P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k)/dt$, it follows from 1·321 and 1·322 that this derivative is multilinear in $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_k$, and dP/dt is *defined* as the core of $d(P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k)/dt$.

1·62. It is on the assumption that the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_k$ not only can be but *are* independent of t that $d(P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k)/dt$ is multilinear and introduces dP/dt. But this derived core is of no less service in the evaluation of $d(P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k)/dt$ when the vectors vary with t, the symbols, in consequence of 1·323 and the definitions, grouping themselves in the familiar manner

1·621 $$d(P\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k)/dt = (dP/dt)\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k + P(d\mathbf{r}_1/dt)\mathbf{r}_2\ldots\mathbf{r}_k + P\mathbf{r}_1(d\mathbf{r}_2/dt)\ldots\mathbf{r}_k + \ldots + P\mathbf{r}_1\mathbf{r}_2\ldots(d\mathbf{r}_k/dt).$$

This identity is sometimes of service for the calculation of $(dP/dt)\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_k$, but there is nothing in the formula so used to shew why the function obtained is multilinear.

1·71. The multilinear functions of differential geometry are not so much functions of directions in space as functions of directions at a point; in other words, they are functions of direction with cores depending on a variable point.

Let $P^k \mathbf{r}_1 \mathbf{r}_2 \ldots \mathbf{r}_k$ be such a function, dependent on the position of a point Q, and suppose Q to be confined to a curve through a particular point O. On this curve P^k can be regarded as a function of the arc s measured to Q from some fixed point, and to calculate the rate of change dP^k/ds a frame of reference may be used; then

1·711 $$\frac{dP^k}{ds} = \frac{\partial P^k}{\partial x}\frac{dx}{ds} + \frac{\partial P^k}{\partial y}\frac{dy}{ds} + \frac{\partial P^k}{\partial z}\frac{dz}{ds},$$

that is,

1·712 $$dP^k/ds = P_1{}^k x_T + P_2{}^k y_T + P_3{}^k z_T,$$

where $P_1{}^k$, $P_2{}^k$, $P_3{}^k$ are functions of position having no relation to the curve described by Q and x_T, y_T, z_T are the direction ratios of the tangent to this curve. Hence

1·713. *The rate at which the core of a multilinear function dependent on position changes at a point O with respect to the arc of a curve through O is the same for all curves whose direction at O is the same,* and can be called simply the rate of change in the common direction, and further

1·714. *The rate of change of the core of a multilinear function in a variable direction is a linear function of that direction.*

If the rate of change of the core P^k in the direction OR is dP^k/ds_R, the function $r(dP^k/ds_R)\mathbf{r}_1 \mathbf{r}_2 \ldots \mathbf{r}_k$ is linear in the vector r_R as well as in the k vectors $\mathbf{r}_1$, $\mathbf{r}_2$, ... $\mathbf{r}_k$ and is therefore a multilinear function of degree $k+1$; its core, which depends only on the variation of P^k in space, is called the *gradient* of P^k and denoted by P^{k+1}. Sometimes the *function* $P^{k+1}\mathbf{r}_1 \mathbf{r}_2 \ldots \mathbf{r}_k \mathbf{r}_{k+1}$ is called the gradient of the *function* $P^k \mathbf{r}_1 \mathbf{r}_2 \ldots \mathbf{r}_k$.

That *linear* and *bilinear* functions have a part to play follows from 1·511 and 1·512, and on account of 1·714 the appearance of functions of higher degrees is inevitable, but it is not every useful multilinear function that is derivable from some linear or bilinear function by the formation of successive gradients.

1·72. If $\mathbf{r}_1$, $\mathbf{r}_2$, ... $\mathbf{r}_k$ instead of being independent variables are definite functions of the position of the current point O on a curve,

the rate of change of the multilinear function $P^k \mathbf{r}_1 \mathbf{r}_2 \ldots \mathbf{r}_k$ with respect to a parameter t on the curve is given by

1·721 $d(P^k \mathbf{r}_1 \mathbf{r}_2 \ldots \mathbf{r}_k)/dt = P^{k+1} \mathbf{r}_1 \mathbf{r}_2 \ldots \mathbf{r}_k \mathbf{w}$

$+ P^k (d\mathbf{r}_1/dt)\, \mathbf{r}_2 \ldots \mathbf{r}_k + P^k \mathbf{r}_1 (d\mathbf{r}_2/dt) \ldots \mathbf{r}_k + \ldots + P^k \mathbf{r}_1 \mathbf{r}_2 \ldots (d\mathbf{r}_k/dt)$,

where $\mathbf{w}$ is the velocity of O with respect to t. In particular, the rate of change of a function of direction $P^k_{AB..K}$ along a curve in the direction OL is given by

1·722 $$dP^k_{AB\,.\,K}/ds_L = P^{k+1}_{AB\;.\,KL} + P^k_{*B\;\;K}(d\mathbf{1}_A/ds_L) + P^k_{A*\;\;K}(d\mathbf{1}_B/ds_L) + \ldots + P^k_{AB\;.\,*}(d\mathbf{1}_K/ds_L);$$

the vector $d\mathbf{1}_H/ds_L$ is not as a rule a unit vector, nor is this vector a linear function of OL unless the direction OH is independent of the direction OL, so that in the majority of applications it is only the first of the terms on the right of 1·722 that is itself a function of direction, but if $d\mathbf{1}_H/ds_L$ can be put into the form $p_P + q_Q + r_R$, where OP, OQ, OR are known directions, then

1·723
$$P^k_{AB..*\;\;K}(d\mathbf{1}_H/ds_L) = pP^k_{AB..\,P\,.\,K} + qP^k_{AB.\;Q.\;K} + rP^k_{AB\;\;R\;.K},$$

and the multilinear function of direction whose core is P^k reappears with different sets of directions for arguments and with scalar multipliers that do not depend on P^k.

1·81. If the source of the linear function Q_T of direction in a plane is the vector r_R, then

1·811 $$Q_T = r \cos \epsilon_{RT}$$

and therefore

$$da\, Q_T = r \cos(\epsilon_{RT} + \tfrac{1}{2}\pi) = r \cos \epsilon_{RE},$$

that is,

1·812 $$da\, Q_T = Q_E:$$

The value of the angular derivative of a linear function in the direction OT is the same as the value of the function itself in the direction which makes a positive right angle with OT.

In other words

1·813. *The angular derivative of a linear function Q_T of direction in a plane is the linear function whose source is obtained by rotating the source of Q_T through a positive right angle.*

1·82. The partial angular derivatives of a multilinear function of independent directions in a plane are given at once by 1·812, and

extension to functions multilinear in interdependent directions is made by the use of 0·62 and 0·63. For example, if P_{ST} is bilinear,

1·821 $$da_S P_{ST} = P_{DT}, \quad da_T P_{ST} = P_{SE},$$

1·822 $$da P_{TT} = P_{ET} + P_{TE} = -\,da P_{EE},$$

1·823 $$da P_{TE} = P_{EE} - P_{TT} = da P_{ET};$$

from 1·822,

1·824. *If P_{ST} is any bilinear function of directions in a plane, the sum of the values of the quadratic function P_{TT} for two directions at right angles is constant;*

1·823 shews that $P_{TE} - P_{ET}$ also is constant, but this is merely a second version of the same theorem, obtained by regarding P_{SE} as a function of OS and OT.

1·83. To look for the angular derivative in an arbitrary plane of a scalar linear function of direction in space is to reach familiar ground. Let R_T be the linear function whose source is $\mathbf{r}$, let ON be any direction in space, and let $\mathbf{s}$ be the component of $\mathbf{r}$ at right angles to ON; if OT is a direction at right angles to ON, the projection of $\mathbf{r}$ on OT is the projection of $\mathbf{s}$ on OT, and therefore the source of R_T in the plane at right angles to ON is $\mathbf{s}$; it follows from 1·813 that the angular derivative of R_T in this plane has for its source the vector obtained by rotating $\mathbf{s}$ through a positive right angle round ON, and this we recognise as the vector product of $\mathbf{r}$ and 1_N.

2. Fundamental Notions in the Kinematical Geometry of Surfaces and Families of Surfaces

2·0. To prepare for geometrical applications of the theory of multilinear functions it is necessary to examine the different vectors of the form $d1_H/ds_K$, where each of the directions OH, OK is either constantly normal or constantly tangential to a definite surface through O, and the rate of change is with respect to the arc of some curve whose direction at O is OK. It is assumed that by a satisfactory convention one of the directions at right angles to the surface is chosen to be called *the* normal direction, and that there is a spatial convention by which the choice of the normal direction determines the direction of angular measurement in the tangent plane at O. The normal direction is denoted by ON, and OR, OS,

OT will be used for arbitrary tangential directions; for the rest, the notation is that described in 0·5.

2·11. The vector $d\mathbf{1}_N/ds_T$ is the velocity of the *Gaussian image* of O as O moves along the surface in the direction OT; it is at right angles to ON, and may be described either directly as due to the spin of the tangent plane about the conjugate tangent or by components or projections with respect to given tangential directions. The latter course has the advantages of involving no difficulties of sign and of introducing two functions of prime importance: if the velocity of the Gaussian image is resolved into a component along the tangent and a perpendicular component, the amount of the first of these in the direction OT' reverse to OT is the *normal curvature* of the surface in the direction OT, and will be denoted by κ_n, and the amount of the second in the direction OE' with which OT makes a positive right angle is the *geodesic torsion* along OT, for which ς_g will be used. Symbolically

2·111 $$d\mathbf{1}_N/ds_T = -\kappa_n \mathbf{1}_T - \varsigma_g \mathbf{1}_E,$$

and since the directions OT, OE are perpendicular

2·112 $$\kappa_n = -\mathscr{S}(d\mathbf{1}_N/ds_T)\,\mathbf{1}_T,$$

2·113 $$\varsigma_g = -\mathscr{S}(d\mathbf{1}_N/ds_T)\,\mathbf{1}_E.$$

• **2·12.** Because the direction ON depends only on the position of O, not on the direction OT, the vector $d\mathbf{1}_N/ds_T$ is a linear function of the direction OT, and the projection of this vector in a direction OS independent of OT is linear in both OS and OT. It follows that

2·121. *The normal curvature and the geodesic torsion of a surface are quadratic functions of direction,*

from which it is a corollary that

2·122. *Neither κ_n nor ς_g can vanish along more than two tangents at O without vanishing in every direction through O.*

2·21. Analysis of the vector $d\mathbf{1}_S/ds_T$ for an *arbitrary* relation of the tangential direction OS to the curve described by O is illuminated by the corresponding analysis of the particular vector $d\mathbf{1}_T/ds_T$. This latter is the vector of curvature of the curve, and being necessarily at right angles to OT is determined by its projections in any two directions normal to the curve. When the curve is being considered in relation to a surface on which it lies, the directions on which the

vector of curvature is projected are ON, the normal to the surface, and OE, the tangential normal to the curve: the amount of the normal projection is the normal curvature κ_n, and the formula

2·211 $$\kappa_n = \mathscr{S}(d\mathbf{1}_T/ds_T)\,\mathbf{1}_N$$

is reconciled with 2·112 by the consideration that since $\mathscr{S}\mathbf{1}_T\mathbf{1}_N$ is constant the sum $\mathscr{S}(d\mathbf{1}_T/dt)\,\mathbf{1}_N + \mathscr{S}(d\mathbf{1}_N/dt)\,\mathbf{1}_T$ is zero whatever the variable t; the amount of the tangential projection is the *geodesic curvature* of the curve, and this will be denoted by κ_g:

2·212 $$\kappa_g = \mathscr{S}(d\mathbf{1}_T/ds_T)\,\mathbf{1}_E.$$

Because ON and OE are at right angles, and coplanar with the vector of curvature,

2·213 $$d\mathbf{1}_T/ds_T = \kappa_n\mathbf{1}_N + \kappa_g\mathbf{1}_E.$$

2·22. The change in a tangential radial $\mathbf{1}_S$ as O moves on the surface is partly a motion *with* the current tangent plane, and partly a motion *in* this plane; the two components, of which the first is wholly normal and the second wholly tangential, play equally useful but dissimilar parts, and have no analytical resemblance.

2·31. The component of $d\mathbf{1}_S/ds_T$ normal to the surface to which OS and OT are tangential I propose to call, for reasons that will become apparent, the *bilinear curvature* of the surface in the directions OS, OT, and to denote by κ_{ST}:

2·311 $$\kappa_{ST} = \mathscr{S}(d\mathbf{1}_S/ds_T)\,\mathbf{1}_N.$$

This function must be recognised in a variety of different forms which are readily found.

The motion of $\mathbf{1}_S$ with the tangent plane is determined by the spin of this plane, which if OC is a direction conjugate to OT is a spin of a definite amount p about OC:

2·312. *If OC is a direction conjugate to OT and the spin of the surface along OT is of amount p round OC, then*

$$\kappa_{ST} = p\sin\epsilon_{CS}.$$

To avoid the use of p, which cannot be made a single-valued function of position and direction by any satisfactory convention, all that is necessary is to resolve the vector p_C along *determinate* directions.

If p_C is resolved into a vector along OS and a perpendicular vector, only the second of these components affects $\mathbf{1}_S$, and the rate of change of $\mathbf{1}_S$ as far as it is due to this component has the same

amount in the direction ON as the component itself has in the direction OD' *with which* OS makes a positive right angle round ON:

2·313. *The bilinear curvature* κ_{ST} *is the projection in the direction with which OS makes a positive right angle of the spin of the current tangent plane as O moves in the direction OT.*

Another aspect is presented if the spin of the tangent plane is related to the velocity of the Gaussian image; the latter of these vectors is obtained by rotating the former through a negative right angle in the tangent plane and therefore

2·314. *The bilinear curvature* κ_{ST} *is the projection in the direction reverse to OS of the velocity of the Gaussian image of O with respect to the arc of any curve in the direction OT.*

This result can be expressed in symbols in the form

2·315 $$\kappa_{ST} = -\mathscr{S}(d\mathbf{1}_N/ds_T)\,\mathbf{1}_S,$$

and is deducible algebraically from the definition 2·311, for since $\mathscr{S}\mathbf{1}_S\mathbf{1}_N$ is always zero,

2·316. $$\mathscr{S}(d\mathbf{1}_S/dt)\,\mathbf{1}_N + \mathscr{S}(d\mathbf{1}_N/dt)\,\mathbf{1}_S = 0,$$

whatever the variable t.

2·32. The relation of bilinear curvature to normal curvature is seen immediately from 2·211 and 2·311:

2·321. *The bilinear curvature of a surface reduces to the normal curvature when the directions on which it depends coincide.*

But the part to be played by the bilinear curvature in coordinating properties of different functions of a single direction is better appreciated after a comparison of 2·315 with 2·112 and 2·113; the identity of κ_n with κ_{TT} appears again, and ς_g is seen to be κ_{ET}:

2·322. *If OE is the tangential direction making a positive right angle with OT, the bilinear curvature* κ_{ET} *is the geodesic torsion of the surface along OT.*

In virtue of 2·321 and 2·322, 2·111 may be written

2·323 $$d\mathbf{1}_N/ds_T = -\kappa_{TT}\mathbf{1}_T - \kappa_{ET}\mathbf{1}_E,$$

and it follows that if $P\mathbf{r}$ is any linear function of a vector,

2·324 $$P(d\mathbf{1}_N/ds_T) = -\kappa_{TT}P_T - \kappa_{ET}P_E = -\kappa_{*T}P_*,$$

because OT and OE are at right angles.

2·33. The apparent duplicity of 2·312 has been removed in 2·313 by means of the definite directions OS and OD; it may be removed

otherwise by the use of OT and OE: since the spin along OT is the sum of κ_n about OE' and ς_g about OT,

2·331 $$\kappa_{ST} = \kappa_n \cos \epsilon_{ST} - \varsigma_g \sin \epsilon_{ST},$$

a formula which in the form

2·332 $$\kappa_{ST} = \kappa_{TT} \sin \epsilon_{SE} + \kappa_{ET} \sin \epsilon_{TS}$$

merely expresses the linearity of the function in the direction OS.

2·41. The tangential component of $d1_S/ds_T$ being necessarily at right angles to OS, its direction of measurement can be chosen and an unambiguous scalar obtained; the amount of the tangential component in the direction which makes a positive right angle with OS I call the *swerve* of OS along OT and denote by $\sigma_T{}^S$, or by σ^S only if the manner of the dependence on OT can be assumed:

2·411 $$\sigma_T{}^S = \mathscr{I}(d1_S/ds_T)\, 1_D.$$

The swerve of OS along OT is the rate at which OS rotates about ON as O moves in the direction of OT; hence if OR, OS are any two tangential directions dependent on the position of O,

2·412 $$\sigma_T{}^S - \sigma_T{}^R = d\epsilon_{RS}/ds_T:$$

The swerve of OS in any direction exceeds the swerve of OR in the same direction by the rate of change of an angle from OR to OS.

From this theorem comes a method of evaluating σ^S by means of a curve in the direction of OT, for from 2·212 and 2·411 it follows that $\sigma_T{}^T$ is κ_g, that is, that

2·413. *If OT is the current tangent to a curve on a surface the swerve of OT along OT is the geodesic curvature of the curve,*

and therefore

2·414 $$\sigma^S = \kappa_g + (d\epsilon_{TS}/ds).$$

The swerve σ^S is equal to κ_g if ϵ_{TS} has *any* constant value, and in particular

2·415 $$\sigma_T{}^E = \kappa_g.$$

2·42. If the direction OS depends only on the position of O, the vector $d1_S/ds_T$ is a linear function of OT, and therefore since the swerve is the projection of this vector *in a direction independent of OT*,

2·421. *The swerve along OT of a tangential direction which depends only on the position of O is a linear function of OT.*

Hence

2·422 $$\sigma_T{}^S \sin\omega = \sigma_A{}^S \sin\beta + \sigma_B{}^S \sin\alpha,$$

and so in particular

2·423 $$\sigma_T{}^T \sin\omega = \sigma_A{}^T \sin\beta + \sigma_B{}^T \sin\alpha,$$

2·424 $$\begin{bmatrix} \sigma_T{}^A \sin\omega = \sigma_A{}^A \sin\beta + \sigma_B{}^A \sin\alpha, \\ \sigma_T{}^B \sin\omega = \sigma_A{}^B \sin\beta + \sigma_B{}^B \sin\alpha, \end{bmatrix}$$

formulae which by 2·414 are equivalent to

2·425 $$\kappa_g \sin\omega = \{\dot\kappa_g + (d\alpha/d\dot s)\} \sin\beta + \{\ddot\kappa_g - (d\beta/d\ddot s)\} \sin\alpha,$$

2·426 $$\begin{bmatrix} \{\kappa_g - (d\alpha/ds)\} \sin\omega = \dot\kappa_g \sin\beta + \{\ddot\kappa_g - (d\omega/d\ddot s)\} \sin\alpha, \\ \{\kappa_g + (d\beta/ds)\} \sin\omega = \{\dot\kappa_g + (d\omega/d\dot s)\} \sin\beta + \ddot\kappa_g \sin\alpha, \end{bmatrix}$$

where $\dot\kappa_g$, $\ddot\kappa_g$ are the geodesic curvatures of the curves of reference and $\dot s$, $\ddot s$ are arcs of these curves. The last two formulae can be used for isolated curves, but 2·425 supposes OT to be known not merely along a particular curve but along the reference curves also, and is therefore available only in the discussion of the typical member of a family of curves; in other words, 2·425 assumes a definite tangential direction to be associated with every point on the surface and gives the geodesic curvature at O of the particular curve which passes through O and has at every one of its points the direction corresponding to that point.

2·51. The definitions 2·311, 2·411 are combined in the equation

2·511 $$d\mathbf{1}_S/ds_T = \kappa_{ST}\mathbf{1}_N + \sigma_T{}^S \mathbf{1}_D,$$

which has for particular cases 2·213 and

2·512 $$d\mathbf{1}_E/ds_T = \varsigma_g \mathbf{1}_N - \kappa_g \mathbf{1}_T.$$

The three formulae 2·213, 2·512, 2·111 express that

2·513. *The frame OTEN has the spins* ς_g, $-\kappa_n$, κ_g;

the calculation of the vector $d\mathbf{1}_S/ds_T$ by means of this moving frame reproduces 2·511, if κ_{ST} and $\sigma_T{}^S$ are regarded as *defined* by 2·331 and 2·414.

2·52. To the first writers on differential geometry, to associate the curvatures and torsions of curves on a surface with the form of the surface itself was the fundamental problem, and if the problem has lost its interest with its difficulties, the solution is not the less valuable. Supposing a curve and its tangential indicatrix both to be free from stationary points, a choice of direction along the principal

normal at a single point fixes the standard direction OP along the current principal normal everywhere, and renders determinate the binormal direction OB and the sign* of the curvature. The fundamental trirectal $OTPB$ has no spin about OP, and its spins about OB and OT are the curvature κ and the torsion ς of the curve.

If a curve is on a given surface, a continuously varying angle, determined by choice at a single point, from OP, the principal normal of the curve, to ON, the normal to the surface, is called the normal angle of the curve on the surface and denoted by ϖ. The spin of the trirectal $OTEN$ differs from that of the trirectal $OTPB$ only by the addition of a component of amount $d\varpi/ds$ about OT; hence the spin of $OTEN$ is compounded of $\varsigma+(d\varpi/ds)$ about OT and κ about OB, and since the latter of these components is the sum of $\kappa\cos\varpi$ about OE' and $\kappa\sin\varpi$ about ON, 2·513 shews that

2·521. *The normal curvature, the geodesic curvature, and the geodesic torsion, of a curve on a surface are related to the curvature and torsion of the curve in space by the formulae*

$$\kappa_n=\kappa\cos\varpi,\quad \kappa_g=\kappa\sin\varpi,\quad \varsigma_g=\varsigma+(d\varpi/ds),$$

where ϖ is the normal angle.

2·61. In dealing with a family of surfaces it is necessary to contemplate the variation of normal and tangential radials when the current point is no longer confined to a single surface. Since a rate of change in any oblique direction can be calculated by means of normal and tangential rates of change, the rates of change that have now to be discussed are normal, that is, are rates of change as the current point describes an orthogonal trajectory of the family, and the arc of this curve will be denoted by n. The vectors to be examined have the forms $d\mathbf{1}_S/dn$ and $d\mathbf{1}_N/dn$.

2·62. To suggest the evaluation of $d\mathbf{1}_S/dn$ presupposes that along the particular orthogonal trajectory under consideration there is associated with each position of O a definite direction OS tangential to the surface through O; the vector $d\mathbf{1}_S/dn$ is then a vector in the plane ODN and is naturally described by its projections on OD and ON. The vector $d\mathbf{1}_N/dn$ is the vector of curvature of the trajectory

* The common convention that in solid geometry this sign must be positive is mischievous beyond words. The curvature of a curve is in fact the amount of a vector, positive if measured in one direction and negative if measured in the reverse.

and is not itself dependent on particular tangential directions, but to describe it by means of scalars reference to specific directions must be made; by a choice of tangential directions having intrinsic relations to the surface a purely intrinsic account of $d\mathbf{1}_N/dn$ can be given, but not only do applications involve the projection of $d\mathbf{1}_N/dn$ on an arbitrary tangential direction OS, but since the directions OS, ON are at right angles, this projection is the negative of the projection $\mathcal{G}(d\mathbf{1}_S/dn)\,\mathbf{1}_N$ which is in any case required in connection with $d\mathbf{1}_S/dn$.

2·71. The tangential component of $d\mathbf{1}_S/dn$, which I call the *swing* of OS round ON, is related to ON just as the swerve of OS along OT is related to OT, and the notation of 2·41 can be adopted:

2·711 $$\sigma_N{}^S = \mathcal{G}(d\mathbf{1}_S/dn)\,\mathbf{1}_D.$$

In fact if OS and OD are directions depending only on the position of O, the projection $\mathcal{G}(d\mathbf{1}_S/ds_P)\,\mathbf{1}_D$ has the same value for all curves in the direction OP, whether this direction is tangential, oblique, or normal, and the function $\sigma_P{}^S$ defined by

2·712 $$\sigma_P{}^S = \mathcal{G}(d\mathbf{1}_S/ds_P)\,\mathbf{1}_D$$

is a linear function of OP.

The result expressed by 2·412 is true whatever the direction of the curve involved, and in particular

2·713 $$\sigma_N{}^S - \sigma_N{}^R = d\epsilon_{RS}/dn,$$

so that

2·714. *If the angle between two tangential directions is constant along a trajectory the directions have the same swing about the normal.*

2·72. To use 2·713 for the calculation of swings, the swing of some one direction must be known. Anticipating acquaintance with the principal tangents of a surface, we observe that because these tangents are at right angles on every surface, the four principal directions have the same swing; this swing I call the *twist* of the family and denote by ϖ. From 2·713,

2·721 $$\sigma_N{}^S = \varpi + (d\zeta/dn),$$

where ζ is an angle to OS from a principal direction of the surface; this formula breaks down at an umbilic, and is quite useless if the family is composed of planes or spheres, when the principal directions

are everywhere indeterminate, but in general the twist is the first swing to be calculated.

Referring for a moment to a topic less elementary than will occupy us in these pages, it may be mentioned that the vanishing of the twist is the necessary and sufficient condition for a family not composed of planes or spheres to be a Lamé family, that is, to be one of three families forming a triply orthogonal system.

2·73. We shall write

2·731 $$\tau_S = \mathscr{S}(d\mathbf{1}_S/dn)\,\mathbf{1}_N,$$

and call the function τ_S the *spread* of the family along OS. Being a linear function of OS, the spread is given with reference to any two tangential directions OA, OB by a formula of the usual type:

2·732 $$\tau_S \sin\omega = \tau_A \sin\beta_S + \tau_B \sin\alpha_S.$$

2·74. Combining 2·731 with 2·711, and noting that $d\mathbf{1}_S/dn$ is necessarily at right angles to $\mathbf{1}_S$, we have

2·741 $$d\mathbf{1}_S/dn = \sigma_N{}^S\,\mathbf{1}_D + \tau_S\,\mathbf{1}_N.$$

2·81. As has been perceived in 2·62,

2·811 $$\tau_S = -\,\mathscr{S}(d\mathbf{1}_N/dn)\,\mathbf{1}_S.$$

The tangential vector $d\mathbf{1}_N/dn$ can be expressed by its projections on any two tangential directions:

$$(d\mathbf{1}_N/dn)\sin\epsilon_{ST} = \{\mathscr{S}(d\mathbf{1}_N/dn)\,\mathbf{1}_D\}\,\mathbf{1}_T - \{\mathscr{S}(d\mathbf{1}_N/dn)\,\mathbf{1}_E\}\,\mathbf{1}_S,$$

that is,

2·812 $$(d\mathbf{1}_N/dn)\sin\epsilon_{ST} = \tau_E\,\mathbf{1}_S - \tau_D\,\mathbf{1}_T.$$

In particular

2·813 $$d\mathbf{1}_N/dn = -\,\tau_S\,\mathbf{1}_S - \tau_D\,\mathbf{1}_D,$$

which combines with 2·741 to express that

2·814. *With respect to the arc of the orthogonal trajectory, the frame OSDN has spins* τ_D, $-\,\tau_S$, $\sigma_N{}^S$.

With 2·741 and 2·813 can therefore be associated

2·815 $$d\mathbf{1}_D/dn = -\,\sigma_N{}^S\,\mathbf{1}_S + \tau_D\,\mathbf{1}_N,$$

but this is only another version of 2·741, for $\sigma_N{}^D$ has the same value as $\sigma_N{}^S$, and OS makes a negative right angle with OD.

2·82. That $d\mathbf{1}_N/dn$ is the vector of curvature of the trajectory

must not be overlooked. Formulae giving the curvature in terms of spreads are

2·821 $$\kappa^2 \sin^2 \epsilon_{ST} = \tau_S^2 - 2\tau_S\tau_T \cos \epsilon_{ST} + \tau_T^2,$$

which is general, and

2·822 $$\kappa^2 = \tau_S^2 + \tau_D^2,$$

where OD and OS are as usual perpendicular.

2·83. Comparison of 2·814 with 2·513 suggests a valuable outlook on the functions $\sigma_N{}^S$, τ_S. Suppose a surface drawn to contain the trajectory under consideration and to have OS for a tangential direction at every point of this curve; ONS is the tangent plane to this surface at O, and if OD is taken for the positive normal direction, the relation of the frame $ONSD$ to the trajectory regarded as a curve on this surface shews that

2·831. *On any surface containing the orthogonal trajectory and having OD for current normal along the trajectory, this curve has geodesic torsion $\sigma_N{}^S$, normal curvature $-\tau_D$, and geodesic curvature $-\tau_S$.*

3. Surfaces and Multilinear Functions associated with a Function of Position in Space

3·11. Referred to a frame $OABC$, a function Φ of position in space becomes a function of the coordinates x, y, z of the variable point, and in all that follows it is assumed that the functions concerned are not merely absolute constants, and are regular.

If Φ_Q denotes the value of Φ at the point Q, the aggregate of points for which Φ has the particular value Φ_Q is the class of points satisfying the equation

3·111 $$\Phi(x, y, z) = \Phi_Q,$$

and is therefore in general a *surface*, the Φ-surface through Q. Singular points are omitted, and the region considered is one throughout which the Φ-surfaces compose a family of which one and only one member passes through any point.

Conversely, any one surface is given by a set of equations of the form

3·112 $$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v),$$

and any family by a set of the same form in which the functions involve in addition to u and v a parametric variable w. The eliminant of u and v from the set of equations 3·112 is a relation between x, y, z, and w which within a sufficiently restricted domain can be put into the form

3·113 $$\Phi(x, y, z) = w.$$

Hence geometrical properties of a Φ-surface and a Φ-family, in so far as they do not involve the function Φ itself, are properties of all regular surfaces and families of surfaces.

3·21. Along a curve, defined by the expression of x, y, z as functions of the arc s, the function Φ has a rate of change given by

3·211 $$\frac{d\Phi}{ds} = \frac{\partial\Phi}{\partial x}\frac{dx}{ds} + \frac{\partial\Phi}{\partial y}\frac{dy}{ds} + \frac{\partial\Phi}{\partial z}\frac{dz}{ds},$$

that is, by

3·212 $$d\Phi/ds = \Phi_x x_T + \Phi_y y_T + \Phi_z z_T,$$

where Φ_x, Φ_y, Φ_z, the partial derivatives of Φ, are themselves functions of position having no relation to the curve, while x_T, y_T, z_T are the ratios of the direction OT of the curve. Thus

3·213. *The rate of change of a regular scalar function of position in space along any curve depends only on the direction of the curve and is a linear function of that direction.*

The linear function whose value in the direction OP is the rate of change of Φ along any curve in that direction will be denoted by $\Phi^1{}_P$, the corresponding function of the vector $\mathbf{p}$ being written $\Phi^1\mathbf{p}$; as with any other linear function,

3·214 $$\Phi^1 p_P = p\Phi^1{}_P$$

and $\Phi^1 1_P$ is identical with $\Phi^1{}_P$.

3·22. The source of the linear function $\Phi^1{}_P$ is called the *gradient** of Φ at O, and will be denoted by $\mathbf{G}$:

3·221 $$\mathscr{S}\mathbf{G}1_P = \Phi^1{}_P.$$

If $\mathbf{G}$ is everywhere the zero vector, then Φ is an absolute constant; this case excepted, the region under consideration, though in special cases it may be broken into a number of separated parts, is not

* It is not necessary to distinguish in practice between the source and the core of a linear function.

sensibly contracted by the omission of the points where **G** is zero. The assumption is therefore made that **G** is nowhere zero, it being understood that the restriction implied is not on Φ but on the domain throughout which results are asserted to hold. Within a united region where **G** is nowhere zero, the two amounts of **G** are separate single-signed functions of position, nowhere zero; one of these functions, not necessarily the one that is positive, is chosen and called the *slope* of Φ; the slope will be denoted by G.

3·23. At a point O where **G** is not zero, the directions in which the rate of change of Φ is zero are the directions at right angles to **G**. Hence

3·231. *The tangent plane at O to the Φ-surface through O is the plane through O at right angles to the gradient of Φ at O,*

and the directions of the normal to the Φ-surface are the directions of **G**; of these directions the one in which **G** has the amount G is determinate, and is called briefly *the* normal direction at O. The normal direction, denoted always by ON, varies regularly with the position of O; hence

3·232. *Every Φ-surface is bifacial within a united region where Φ is regular and the gradient of Φ is nowhere the zero vector,*

and the choice of sign for the slope G determines implicitly the direction of angular measurement in every tangent plane.

3·31. The gradient of the core Φ^1 of the linear function $\Phi^1{}_P$ is denoted by Φ^2, and the bilinear function $\Phi^2{}_{PQ}$ is called the *bilinear rate of change* of Φ in the directions OP, OQ. Differentiation of the sum $\Phi_x x_P + \Phi_y y_P + \Phi_z z_P$ with respect to a variable which is not involved in the ratios x_P, y_P, z_P gives

3·311 $$\Phi^2{}_{PQ} = \Sigma \Phi_{uv} u_P v_Q, \quad u, v = x, y, z,$$

where the summation covers the nine possible terms; since the second derivatives Φ_{uv}, Φ_{vu} are equal,

3·312. *The bilinear rate of change of any regular function is symmetrical in the two directions on which it depends.*

3·32. If the direction OQ coincides with the direction OP, the bilinear function $\Phi^2{}_{PQ}$ becomes a function $\Phi^2{}_{PP}$ which may be called the quadratic rate of change of Φ in the direction OP. This function must not be confused with the second order rate of change $d^2\Phi/ds_P{}^2$,

which is not the same for all curves in the direction OP: application of 1·722 to

3·321 $$d\Phi/ds_P = \Phi^1{}_P$$

gives

3·322 $$d^2\Phi/ds_P{}^2 = \Phi^2{}_{PP} + \Phi^1(d\mathbf{1}_P/ds_P) = \Phi^2{}_{PP} + \mathscr{G}\mathbf{G}(d\mathbf{1}_P/ds_P),$$

and since $d\mathbf{1}_P/ds_P$ is the vector of curvature of the particular curve along which the rate of change is being found, it is only when either the curvature is zero or the principal normal is tangential to the Φ-surface that the last term disappears from 3·322.

3·41. The conception of the bilinear rate of change, and the fundamental theorem 3·312, are immediately extended. The core Φ^2 has a gradient Φ^3, and so on, and the multilinear rate of change of Φ of degree k is the function $\Phi^k{}_{PQ\ \ V}$, where for each value of h in succession Φ^{h+1} is the gradient of Φ^h. With a frame of reference,

3·411 $$\Phi^k{}_{PQ\ \ V} = \Sigma\Phi_{st.\ w}s_P t_Q \dots w_V, \quad s, t, \dots w = x, y, z,$$

the coefficients being the partial derivatives when Φ is expressed as a function of x, y, z; hence

3·412. *Every multilinear rate of change of a regular function of position is symmetrical in the variable directions.*

3·51. The rate of change of Φ along any curve on a Φ-surface being zero,

3·511 $$\Phi^1{}_S = 0$$

if OS is restricted as usual to denote a direction tangential to the Φ-surface at O; on the other hand by the definition of the slope

3·512 $$\Phi^1{}_N = G.$$

From 3·511 and 3·512 together comes the expression for $\Phi^1\mathbf{r}$ when $\mathbf{r}$ is arbitrary: if $\mathbf{r}$ is expressed as $p_S + q_N$ where OS is tangential, then because the function $\Phi^1\mathbf{r}$ is linear

3·513 $$\Phi^1\mathbf{r} = p\Phi^1{}_S + q\Phi^1{}_N,$$

and substitution from 3·511 and 3·512 gives

3·514 $$\Phi^1\mathbf{r} = Gq,$$

that is,

3·515 $$\Phi^1\mathbf{r} = G\mathscr{G}\,\mathbf{r}\mathbf{1}_N;$$

this formula is of course obvious from the definition of $\mathbf{G}$.

4. The Bilinear Curvature of a Surface

4·11. That

4·111. *The bilinear curvature of a surface is a bilinear function of the two tangential directions on which it depends*

is obvious equally in every expression given for κ_{ST} in 2·3, and this property alone implies such formulae as

4·112 $$\kappa_{ST}\sin^2\omega = \kappa_{AA}\sin\beta_S\sin\beta_T + \kappa_{AB}\sin\beta_S\sin\alpha_T + \kappa_{BA}\sin\alpha_S\sin\beta_T + \kappa_{BB}\sin\alpha_S\sin\alpha_T,$$

4·113 $$da_S\kappa_{ST} = \kappa_{DT}, \quad da_T\kappa_{ST} = \kappa_{SE},$$

where OS, OT are independent of each other,

4·114 $$da_R\kappa_{ST} = (1 + da_R\epsilon_{RS})\,\kappa_{DT} + (1 + da_R\epsilon_{RT})\,\kappa_{SE},$$

if OS, OT depend on a tangential direction OR, and in particular

4·115 $$da\,\kappa_{TT} = \kappa_{ET} + \kappa_{TE} = -\,da\,\kappa_{EE},$$

4·116 $$da\,\kappa_{TE} = \kappa_{EE} - \kappa_{TT} = da\,\kappa_{ET},$$

special cases of 1·822 and 1·823.

4·12. Dupin's theorem, that

4·121. *At any ordinary point of a surface the sum of the normal curvatures in two directions at right angles is a constant,*

is shewn by 2·321 to be a case of 1·824, that is, to follow from the simple fact that the normal curvature is a quadratic function.

The half of the constant sum $\kappa_{TT} + \kappa_{EE}$ is the *mean curvature* of the surface at O, and will be denoted by B;

4·122 $$\kappa_{TT} + \kappa_{EE} = 2B.$$

The differences $\kappa_n - B$ and $B - \kappa_n$ are the *excess* and the *defect* of curvature along OT. To write

4·123 $$\kappa_{EE} = 2B - \kappa_n$$

is to express κ_{EE} directly as a function of OT, and the function $\kappa_{EE} - \kappa_{TT}$, which appears in 4·116 and in a number of other formulae, is given by

4·124 $$\kappa_{EE} - \kappa_{TT} = 2\,(B - \kappa_n),$$

that is to say, is twice the defect of curvature.

An actual formula giving the mean curvature is easy to find, for 4·112 gives

4·125 $\kappa_{TT}\sin^2\omega = \kappa_{AA}\sin^2\beta + (\kappa_{AB}+\kappa_{BA})\sin\beta\sin\alpha + \kappa_{BB}\sin^2\alpha,$

and substitution of $\alpha+\frac{1}{2}\pi$, $\beta-\frac{1}{2}\pi$ for α, β gives

4·126 $\kappa_{EE}\sin^2\omega = \kappa_{AA}\cos^2\beta - (\kappa_{AB}+\kappa_{BA})\cos\beta\cos\alpha + \kappa_{BB}\cos^2\alpha,$

whence by addition we have not only a trigonometrical proof of Dupin's theorem but the explicit result

4·127 $$2B\sin^2\omega = \kappa_{AA} - (\kappa_{AB}+\kappa_{BA})\cos\omega + \kappa_{BB}.$$

4·21. But bilinearity alone does not account for the importance of the function κ_{ST}. Differentiation of 3·511 along a curve on a Φ-surface gives

4·211 $$\Phi^2{}_{ST} + \Phi^1\,(d1_S/ds_T) = 0,$$

and substituting from 3·515 we see from 2·311 that

4·212. *The bilinear rate of change of a function Φ along two directions OS, OT tangential to the Φ-surface is connected with the bilinear curvature κ_{ST} of the surface in those directions by the equation*

$$\Phi^2{}_{ST} + G\kappa_{ST} = 0,$$

where G is the slope of Φ.

And this result not only enables the bilinear curvature to be calculated in specific cases, but taken with 3·312 shews that

4·213. *At any ordinary point of any surface, the bilinear curvature in two directions is a symmetric function of those directions.*

From the combination of this result with 4·111 springs the whole elementary theory of the curvature of a surface.

4·22. Since identically

4·221 $$\sin\beta_S\sin\alpha_T - \sin\alpha_S\sin\beta_T = \sin\omega\sin\epsilon_{ST},$$

the necessary and sufficient condition for 4·213 to follow from the explicit formula 4·112 is the equality of the coefficients κ_{AB}, κ_{BA}:

4·222. *The symmetry of the bilinear curvature for any one pair of distinct directions at a point implies algebraically the symmetry of this function for any other pair of directions at the same point.*

With the substitution of κ_{AB} for κ_{BA}, 4·112 takes the form

4·223 $$\kappa_{ST}\sin^2\omega = \kappa_{AA}\sin\beta_S\sin\beta_T + \kappa_{AB}(\sin\beta_S\sin\alpha_T + \sin\alpha_S\sin\beta_T) + \kappa_{BB}\sin\alpha_S\sin\alpha_T,$$

giving

$$4{\cdot}224^* \quad \kappa_n \sin^2 \omega = \kappa_{AA} \sin^2 \beta + 2\kappa_{AB} \sin \beta \sin \alpha + \kappa_{BB} \sin^2 \alpha,$$

$$4{\cdot}225 \quad \varsigma_g \sin^2 \omega = -\kappa_{AA} \sin \beta \cos \beta - \kappa_{AB} \sin (\alpha - \beta) + \kappa_{BB} \sin \alpha \cos \alpha,$$

and 4·127 becomes

$$4{\cdot}226 \qquad 2B \sin^2 \omega = \kappa_{AA} - 2\kappa_{AB} \cos \omega + \kappa_{BB}.$$

It need hardly be said that all relations between bilinear curvatures of one surface in different pairs of directions at one point are deducible from 4·223 by pure trigonometry, or that this method of deduction has nothing to recommend it.

4·23. A simple case of 4·213 is the assertion that if κ_{ST}, as described in 2·3, is zero for a particular pair of directions, then so also is κ_{TS}. To say that κ_{ST} is zero is to assert that as O moves in the direction OT the tangent plane at O rotates about OS; in other words

4·231. *A pair of directions for which the bilinear curvature is zero is a pair of conjugate directions.*

Hence 4·213 includes the familiar theorem that

4·232. *If OS is conjugate to OT then OT is conjugate to OS,*
and the application of 4·222 to this result takes the form that

4·233. *If a surface is known to have a single pair of mutually conjugate distinct tangents at a point, the symmetry of the bilinear curvature at that point can be inferred.*

4·24. By means of 2·331 the symmetry of the bilinear curvature can be expressed as a relation between the normal curvatures and geodesic torsions in two directions without explicit mention of the bilinear function; comparing the two formulae

$$4{\cdot}241 \qquad \kappa_{ST} = \kappa_{TT} \cos \epsilon_{ST} - \kappa_{ET} \sin \epsilon_{ST},$$

$$\kappa_{TS} = \kappa_{SS} \cos \epsilon_{ST} + \kappa_{DS} \sin \epsilon_{ST}$$

we have

$$4{\cdot}242 \qquad (\kappa_{TT} - \kappa_{SS}) \cos \epsilon_{ST} = (\kappa_{ET} + \kappa_{DS}) \sin \epsilon_{ST},$$

* This formula shews that a geometrical theory without the bilinear curvature is as incomplete as an analytical theory without the function for which M is used by Scheffers, Forsyth, writers in the *Encyk. d. Math. Wiss.*, and others, D' by Bianchi, and $D'/\sqrt{(EG-F^2)}$ by Gauss and Darboux.

or to use a notation convenient with reference curves

4·243 $$(\dot{\kappa}_n - \dot{\kappa}_n)\cos\omega = (\dot{\varsigma}_g + \dot{\varsigma}_g)\sin\omega,$$

a result given in other symbols and used again and again by Darboux.

4·31. The relation between geodesic torsions in perpendicular directions is simpler in form than the relation between normal curvatures asserted in Dupin's theorem, but belongs in fact to a more advanced stage, depending as it does not on the bilinearity alone but on the symmetry of the bilinear curvature. To write down this relation from 4·225 or 4·243 is of course simple enough, but an appeal to first principles shews more clearly on what the result depends. The *linearity* of κ_{ST} in the direction OT implies

4·311 $$\kappa_{ST'} = -\kappa_{ST},$$

and in virtue of the *symmetry* of the function this equation gives

4·312 $$\kappa_{ST} + \kappa_{T'S} = 0\,;$$

hence in particular

4·313 $$\kappa_{ET} + \kappa_{T'E} = 0,$$

and since OT' is the direction making a *positive* right angle with OE the function $\kappa_{T'E}$ is the geodesic torsion along OE:

4·314. *The sum of the geodesic torsions in two directions at right angles is zero.*

This result, like 4·232, is a special case of 4·213 and implies the more general theorem in which it is included; thus 4·314 and 4·232 in spite of their diversity of form are theorems implying each other, that is, are equivalent theorems, on account of the bilinearity of the bilinear curvature.

4·32. Brevity is often achieved by the use of the function $\frac{1}{2}(\kappa_{DT} + \kappa_{SE})$, which is the symmetrical bilinear function of OS and OT that reduces to the geodesic torsion κ_{ET} when OS coincides with OT; it is natural to write

4·321 $$\varsigma_{ST} = \tfrac{1}{2}(\kappa_{DT} + \kappa_{SE})$$

and to call this function the bilinear torsion, but it must be recognised that the function has none of the fundamental importance of the bilinear curvature. Identically,

4·322 $$\varsigma_{TT} = \kappa_{ET} = \varsigma_g,$$

4·323 $$\varsigma_{ET} = \tfrac{1}{2}(\kappa_{EE} - \kappa_{TT}) = B - \kappa_n,$$

and 4·314 can be expressed in the form

4·324 $$\varsigma_{EE} = -\varsigma_g.$$

Being bilinear and symmetrical, the function ς_{ST} has its value given in terms of directions of reference OA, OB by

4·325 $$\varsigma_{ST}\sin^2\omega = \varsigma_{AA}\sin\beta_S\sin\beta_T + \varsigma_{AB}(\sin\beta_S\sin\alpha_T + \sin\alpha_S\sin\beta_T) + \varsigma_{BB}\sin\alpha_S\sin\alpha_T.$$

But unlike the coefficients κ_{AA}, κ_{AB}, κ_{BB}, the coefficients ς_{AA}, ς_{AB}, ς_{BB} are not numerically independent, for the sum $\varsigma_{TT} + \varsigma_{EE}$ is not merely constant but is zero:

4·326 $$\varsigma_{AA} - 2\varsigma_{AB}\cos\omega + \varsigma_{BB} = 0.$$

4·33. The angular derivatives of the normal curvature and geodesic torsion are given in 4·115 and 4·116. Since κ_{TE} as well as κ_{ET} is ς_g, the first of these formulae becomes

4·331 $$d_a\kappa_n = 2\varsigma_g,$$

a familiar result; 4·116 is equivalent to

4·332 $$d_a\varsigma_g = 2(B - \kappa_n),$$

which is therefore more elementary than 4·331 since it is proved without reference to the symmetry of κ_{ST}. There is a temptation to replace 4·331 by

4·333 $$d_a(\kappa_n - B) = 2\varsigma_g$$

and to treat as correlative the geodesic torsion and the excess of curvature, but the suggested analogy must not be pressed too far.

Written in the forms

$$d_a\kappa_{TT} = 2\varsigma_{TT}, \quad d_a\kappa_{ET} = 2\varsigma_{ET}$$

4·331, 4·332 are seen to be corollaries of the more general theorem that

4·334. *If OS is inclined to OT at a constant angle, then*

$$d_a\kappa_{ST} = 2\varsigma_{ST},$$

an immediate deduction from 4·114.

4·41. A function of direction that is not a mere constant must have at least one direction of maximum value and one of minimum. If OT is a direction along which the value of κ_n is a minimum, then the value along OT' is the same minimum, while it follows from Dupin's theorem that along OE and OE' the value is a maximum. Hence unless κ_n has the same value in every direction

from O, there certainly are two distinct tangents along which the value of κ_n is stationary. On the other hand, 4·331 implies that a tangent along which κ_n is stationary is a tangent along which ς_g is zero, and since ς_g is a quadratic function there cannot be more than two of these tangents unless ς_g is zero in every direction.

4·411. *At an ordinary point of a surface, either the normal curvature is the same in all directions and the geodesic torsion is zero in every direction, or there is one tangent along which the normal curvature has its least value and one along which the normal curvature has its greatest value, these tangents are at right angles and are the only tangents along which the geodesic torsion is zero, and the normal curvature in a variable direction increases or decreases steadily as the direction rotates from one of these tangents to the other.*

A point at which the normal curvature has the same value in all directions is an umbilic; the constant value is of course equal to the mean curvature B at the point.

4·42. From 4·332 and 4·333 it follows that the sum $(\kappa_n - B)^2 + \varsigma_g^2$ does not vary with OT but is a function only of the position of O on the surface, in general positive but zero if and only if O is umbilical. Spheres and planes are composed wholly of umbilics, but from a surface that is neither plane nor spherical the umbilics can be removed, for it can be proved that they are isolated points or compose isolated curves. Throughout a region which is nowhere umbilical, the two square roots of $(\kappa_n - B)^2 + \varsigma_g^2$ are separate single-valued functions of position; one of these, selected and called the *amplitude of curvature*, will be denoted by A:

4·421 $$(\kappa_n - B)^2 + \varsigma_g^2 = A^2.$$

From 4·421 the extreme values of κ_n at a point O, corresponding to the directions along which ς_g is zero, are $B - A$ and $B + A$; these are the *principal curvatures* of the surface at O, and I write

4·422 $$\varkappa_1 = B - A, \quad \varkappa_2 = B + A.$$

The *principal tangents*, that is, the tangents along which the normal curvatures are $\varkappa_1$, $\varkappa_2$, are individually determinate when the sign of A has been chosen. To secure complete freedom from ambiguity, definite directions along these tangents must be chosen also; the choice along one principal tangent at one point is arbi-

trary, and determines the standard direction along the corresponding tangent at all neighbouring points; the standard direction along the other principal tangent is then fixed by the convention that

4·423. *One of the angles from the first principal direction to the second is a* positive *right angle.*

The principal directions at O will be denoted by OC_{I}, $OC_{\mathfrak{z}}$, but as affixes I, $\mathfrak{z}$ will be substituted for C_{I}, $C_{\mathfrak{z}}$.

4·43. The equation

4·431 $$\kappa_{ET} = 0$$

which characterises the directions of curvature implies of course

4·432 $$\kappa_{E'T} = 0,$$

and is therefore equivalent to the combination of

4·433 $$\kappa_{ST} = 0$$

with the condition that OS and OT are at right angles:

4·434. *At any ordinary point that is not an umbilic, the principal tangents are the only two conjugate tangents at right angles.*

4·44. From the definitions and the convention of 4·423,

4·441 $$\kappa_{\mathrm{I}\mathrm{I}} = \varkappa_{\mathrm{I}}, \quad \kappa_{\mathrm{I}\mathfrak{z}} = 0, \quad \kappa_{\mathfrak{z}\mathfrak{z}} = \varkappa_{\mathfrak{z}},$$

4·442 $$\epsilon_{\mathrm{I}\mathfrak{z}} = \tfrac{1}{2}\pi.$$

Substitution in 4·223 gives for any two directions

4·443 $$\kappa_{ST} = \varkappa_{\mathrm{I}} \cos\zeta_S \cos\zeta_T + \varkappa_{\mathfrak{z}} \sin\zeta_S \sin\zeta_T,$$

where ζ denotes an angle to the variable direction from the first principal direction; the forms corresponding to 4·224 and 4·225 which are special cases of 4·443 are

4·444 $$\kappa_n = \varkappa_{\mathrm{I}} \cos^2\zeta + \varkappa_{\mathfrak{z}} \sin^2\zeta,$$

4·445 $$\varsigma_g = (\varkappa_{\mathfrak{z}} - \varkappa_{\mathrm{I}}) \cos\zeta \sin\zeta,$$

the formulae of Euler and Bonnet, of which the first was transformed by Euler himself into the shape

4·446 $$\kappa_n = B - A\cos 2\zeta$$

and the second is

4·447 $$\varsigma_g = A \sin 2\zeta,$$

A, B having the meanings assigned in 4·11 and 4·42. Corollaries of 4·444 are

4·448 $$\kappa_n - \varkappa_{\mathrm{I}} = 2A\sin^2\zeta, \quad \varkappa_{\mathfrak{z}} - \kappa_n = 2A\cos^2\zeta,$$

which with 4·447 give

4·449 $$\varsigma_g^2 = (\varkappa_{\mathfrak{z}} - \kappa_n)(\kappa_n - \varkappa_{\mathfrak{i}}),$$

a relation which is otherwise evident from 4·421.

4·45. To relate 4·443 to the fundamental property of a direction of curvature is a most instructive exercise. If OC is a direction of curvature and $\varkappa$ is the corresponding principal curvature κ_{CC}, the spin along OC is a vector of amount $\varkappa$ in the direction with which OC makes a positive right angle, and therefore the projection of this vector on the direction OE' with which OT makes a positive right angle is $\varkappa \cos \epsilon_{CT}$:

4·451. *If OC is a direction of curvature and $\varkappa$ is the corresponding principal curvature, the bilinear curvature κ_{CT} has the value $\varkappa \cos \epsilon_{CT}$.*

Thus

4·452 $$\kappa_{\mathfrak{i}T} = \varkappa_{\mathfrak{i}} \cos \zeta_T, \quad \kappa_{\mathfrak{z}T} = \varkappa_{\mathfrak{z}} \sin \zeta_T;$$

because κ_{ST} is linear in OS and the principal directions are at right angles,

4·453 $$\kappa_{ST} = \kappa_{\mathfrak{i}T} \cos \zeta_S + \kappa_{\mathfrak{z}T} \sin \zeta_S,$$

and substitution from 4·452 reproduces 4·443.

4·46. Nor is the proof of 4·443 just given the only use, or the chief use, of 4·451; it is from 4·451 that come formulae for determining the principal curvatures and tangents in terms of magnitudes related to arbitrary tangential directions of reference.

Because κ_{AT} and κ_{TB} are linear in OT,

4·461 $$\begin{cases} \kappa_{AT} \sin \omega = \kappa_{AA} \sin \beta + \kappa_{AB} \sin \alpha, \\ \kappa_{TB} \sin \omega = \kappa_{AB} \sin \beta + \kappa_{BB} \sin \alpha, \end{cases}$$

for any tangential direction, while from 4·451

4·462 $$\kappa_{AC} = \varkappa \cos \alpha_C, \quad \kappa_{CB} = \varkappa \cos \beta_C,$$

for a principal direction OC. Hence

4·463. *A direction of curvature in which the normal curvature is $\varkappa$ is characterised by the pair of equations*

$$\begin{cases} \kappa_{AA} \sin \beta + \kappa_{AB} \sin \alpha = \varkappa \sin \omega \cos \alpha, \\ \kappa_{AB} \sin \beta + \kappa_{BB} \sin \alpha = \varkappa \sin \omega \cos \beta. \end{cases}$$

Elimination of $\varkappa$ reproduces the equation obtained more simply by

equating to zero the geodesic torsion as given by 4·225. On the other hand, since identically

$$\sin\omega\cos\alpha = \sin\beta + \cos\omega\sin\alpha, \quad \sin\omega\cos\beta = \sin\alpha + \cos\omega\sin\beta,$$

the equations of 4·463 can be written as

4·464 $$\begin{cases}(\kappa_{AA} - \varkappa)\sin\beta + (\kappa_{AB} - \varkappa\cos\omega)\sin\alpha = 0,\\ (\kappa_{AB} - \varkappa\cos\omega)\sin\beta + (\kappa_{BB} - \varkappa)\sin\alpha = 0.\end{cases}$$

Elimination of the ratio $\sin\beta : \sin\alpha$ yields an equation which $\varkappa$ must satisfy, and since this equation is quadratic, it has no roots except $\varkappa_1$ and $\varkappa_2$:

4·465. *The principal curvatures of a surface are the roots of the equation*

$$(\varkappa - \kappa_{AA})(\varkappa - \kappa_{BB}) = (\varkappa\cos\omega - \kappa_{AB})^2.$$

The equation of 4·465 expands to

4·466 $$\varkappa^2\sin^2\omega - \varkappa(\kappa_{AA} - 2\kappa_{AB}\cos\omega + \kappa_{BB}) + (\kappa_{AA}\kappa_{BB} - \kappa_{AB}^2) = 0,$$

and therefore $2B$, which is the sum of the principal curvatures, and the product of these curvatures, which is the specific or absolute curvature of the surface at O, and is denoted always by K, are given by

4·467 $$2B\sin^2\omega = \kappa_{AA} - 2\kappa_{AB}\cos\omega + \kappa_{BB},$$

which has been obtained already in 4·226, and

4·468 $$K\sin^2\omega = \kappa_{AA}\kappa_{BB} - \kappa_{AB}^2;$$

the amplitude of curvature is determined numerically from the identity

4·469 $$B^2 - A^2 = K.$$

4·47. The fluctuations of the geodesic torsion ς_g are seen most readily from Bonnet's formula 4·447;

4·471. *The extreme values of ς_g are $-A$ and A, and these are assumed in the directions midway between consecutive principal directions.*

The discussion of ς_{TT} as the function defined by identifying OS with OT in 4·325 is parallel to the discussion of κ_{TT} as the function given by identifying OS with OT in 4·223, and therefore the extreme values of ς_{TT} have for their sum $(\varsigma_{AA} - 2\varsigma_{AB}\cos\omega + \varsigma_{BB})\operatorname{cosec}^2\omega$ and

for their product $(\varsigma_{AA}\varsigma_{BB} - \varsigma_{AB}^2)\operatorname{cosec}^2\omega$. Thus 4·326 is reproduced, and the amplitude of curvature is seen to be given by the equation

4·472 $$A^2 \sin^2\omega = \varsigma_{AB}^2 - \varsigma_{AA}\varsigma_{BB}.$$

4·48. If $\sin\alpha$ and $\sin\beta$ are *both* known, the direction OT is determinate, but the *ratio* of the sines alone does not distinguish OT from OT'. Thus in general when $\varkappa$ has a definite one of its possible values, either equation in 4·464 defines the corresponding *principal tangent* but not the corresponding *principal direction*. These formulae however render precise a detail left vague in 4·42. If

4·481 $$\sin\alpha/p = \sin\beta/q = \sin\omega/r,$$

where p, q, r are functions of position on the surface, then

4·482 $$r^2 = p^2 + 2pq\cos\omega + q^2,$$

and throughout a region where p and q do not vanish simultaneously, r is a single-signed function determined everywhere by 4·482 if its sign is known. Hence the choice of sign of a single radical determines the principal direction corresponding to the principal curvature $\varkappa$ throughout the whole of a region provided that no points are included where simultaneously

4·483 $$\kappa_{AA} = \varkappa, \quad \kappa_{AB} = \varkappa\cos\omega, \quad \kappa_{BB} = \varkappa.$$

But

$$\kappa_{AB} = \kappa_{AA}\cos\omega$$

implies that OA is a direction of curvature,

$$\kappa_{AB} = \kappa_{BB}\cos\omega$$

implies that OB is a direction of curvature, and since by hypothesis OA, OB lie along distinct tangents, κ_{AA} and κ_{BB} are the extreme values of the normal curvature, and the additional equality

$$\kappa_{AA} = \kappa_{BB}$$

implies that O is umbilical: having excluded umbilical points for the purpose of separating the principal curvatures, we have actually obtained a region in which the various principal directions also are separated.

4·49. On any surface, a curve whose tangent at every point is a principal tangent of the surface there, or in other words whose geodesic torsion is everywhere zero, is called a line of curvature of the surface. Throughout a united region containing no singular or

umbilical points, the two principal directions at O are definite directions depending regularly on the position of O. It follows from the theory of differential equations that over such a region there are two distinct families of lines of curvature and that through each point passes one and only one member of each family.

If F is a regular function of position of any kind on the surface, the values at O of the rates of change of F in the two positive directions along the two lines of curvature through O depend only on the position of O and define by their relations to O two functions of position which will be denoted by dF/ds_1, dF/ds_2. These functions are not partial derivatives; if in order to use s_1 and s_2 as actual coordinates we go so far as to define the position of O by its distances from two selected trajectories measured along lines of curvature, it is still impossible to secure that every curve along which s_2 has a constant value is itself a line of curvature or has s_1 for its arc; thus even in this case the partial derivative $\partial F/\partial s_1$ is not the rate of change in the direction to which it does correspond and bears no intrinsic relation to dF/ds_1. It follows that although there are rates of change d^2F/ds_1^2, d^2F/ds_2ds_1 derivable from dF/ds_1 and rates of change d^2F/ds_1ds_2, d^2F/ds_2^2 derivable from dF/ds_2, there is no reason to anticipate equality of d^2F/ds_1ds_2 to d^2F/ds_2ds_1; in point of fact it is easy when F is scalar to evaluate the difference between d^2F/ds_1ds_2 and d^2F/ds_2ds_1 and to recognise the rare cases in which this difference vanishes.

4·51. A direction of curvature is a direction in which the geodesic torsion is zero. If there are directions in which the normal curvature is zero, these directions, which are called *asymptotic*, have properties not less interesting than have the directions of curvature.

Since the normal curvature at O varies continuously between its extreme values $\varkappa_1$, $\varkappa_2$, the existence of asymptotic directions depends on the relation between the signs of these two curvatures, that is, depends on the sign of the product K. If K is strictly positive, there are no asymptotic directions and O is said to be an elliptic point on the surface. If K is zero, one if not both of the principal curvatures vanishes, and O is said to be parabolic. For both of the principal curvatures to vanish, that is, for a point to be umbilical as well as parabolic, is altogether exceptional on any surface but a plane. At an ordinary parabolic point, one only of the principal curvatures

vanishes, and the asymptotic directions are the corresponding directions of curvature. A developable is a surface composed wholly of parabolic points, but on a surface that is not developable the parabolic points in general, if there are any, compose a curve or a number of distinct curves separating regions throughout which K is positive from regions throughout which K is negative. In discussing asymptotic directions attention is confined in the first place to a united region composed wholly of hyperbolic points, that is, of points where K is strictly negative.

4·52. Between consecutive directions of curvature at a hyperbolic point, there is one and only one direction in which κ_n, changing in sign from the sign of $\varkappa_1$ to the sign of $\varkappa_2$, is zero; thus there are four distinct asymptotic directions, and since the reverse of an asymptotic direction is itself asymptotic these are the four directions along two asymptotic tangents.

With a direction of curvature is associated the corresponding normal curvature, which is a principal curvature of the surface. In the case of an asymptotic direction it is the geodesic torsion that survives, and this magnitude is called the asymptotic torsion associated with the direction.

The fundamental relation of an asymptotic direction OI to an arbitrary direction OT corresponds to 4·451. The spin of the surface as O moves in the asymptotic direction OI has no component at right angles to OI but is simply a spin of amount ς_{II} about OI, if ς_{II} is the asymptotic torsion along OI; the projection of this spin in the direction OE' is therefore $\varsigma_{II} \sin \epsilon_{IT}$:

4·521. *If OI is an asymptotic direction and ς_{II} is the corresponding asymptotic torsion, the bilinear curvature κ_{IT} has the value $\varsigma_{II} \sin \epsilon_{IT}$.*

If OJ, OK are two asymptotic directions at O, the bilinear curvature κ_{JK} is shewn by 4·521 to be expressible both as $\varsigma_{JJ} \sin \epsilon_{JK}$ and as $\varsigma_{KK} \sin \epsilon_{KJ}$; it follows without reference to the principal directions that

4·522. *The two asymptotic torsions at a hyperbolic point of a surface are equal in magnitude and of opposite sign,*

and it follows also that if the existence of two distinct asymptotic tangents is known 4·522 implies the complete symmetry of the

bilinear curvature. To find the actual values of the asymptotic torsions we have only to compare 4·521 with 4·451: if θ is an angle from OI to the first principal direction, then

4·523 $$\varsigma_{II} \sin\theta = \kappa_{I\iota} = \varkappa_{\iota} \cos\theta,$$

and since $\theta + \frac{1}{2}\pi$ is an angle from OI to the second principal direction,

4·524 $$\varsigma_{II} \cos\theta = \kappa_{I\tau} = -\varkappa_{\tau} \sin\theta;$$

the combination of 4·523 with 4·524 gives

4·525 $$\varsigma_{II}^2 = -\varkappa_{\iota}\varkappa_{\tau},$$

that is,

4·526. *The square of the asymptotic torsions is the negative of the specific curvature,*

a theorem usually ascribed to Enneper but in fact announced by Beltrami four years earlier than by Enneper.

4·53. For the determination of asymptotic directions from arbitrary directions of reference 4·521 is again useful. Comparing

4·531 $$\kappa_{AI} = -\varsigma_{II} \sin\alpha_I, \quad \kappa_{IB} = \varsigma_{II} \sin\beta_I,$$

which are implied by 4·521, with the general formulae 4·461, namely,

4·532 $$\begin{cases} \kappa_{AT} \sin\omega = \kappa_{AA} \sin\beta + \kappa_{AB} \sin\alpha, \\ \kappa_{TB} \sin\omega = \kappa_{AB} \sin\beta + \kappa_{BB} \sin\alpha, \end{cases}$$

we find that

4·533. *An asymptotic direction with asymptotic torsion* ς_{II} *is characterised by the pair of equations*

$$\begin{cases} \kappa_{AA} \sin\beta + (\kappa_{AB} + \varsigma_{II} \sin\omega) \sin\alpha = 0, \\ (\kappa_{AB} - \varsigma_{II} \sin\omega) \sin\beta + \kappa_{BB} \sin\alpha = 0. \end{cases}$$

To eliminate ς_{II} is to obtain the equation expressing that the normal curvature is zero; the elimination of $\sin\beta : \sin\alpha$ gives

4·534 $$\varsigma_{II}^2 \sin^2\omega = \kappa_{AB}^2 - \kappa_{AA}\kappa_{BB},$$

a formula which 4·468 shews to be equivalent to 4·525.

4·54. Throughout a region where K is strictly negative, the asymptotic tangents are distinguished by the asymptotic torsions, which are separate functions of position. One of these square roots of $-K$ is chosen and called the first asymptotic torsion: it will be denoted by $-\varsigma_a$; the second asymptotic torsion is ς_a. Since the four quadrants into which the tangent plane at O is divided by

the principal tangents $C_1'OC_1$, $C_2'OC_2$ are distinct, and the four asymptotic directions lie one in each of these quadrants, the asymptotic directions also are distinct. One of the directions of the first asymptotic tangent, chosen arbitrarily at one point and in consequence determinate elsewhere, is called the first asymptotic direction and denoted by OJ, and an angle from this direction to the first principal direction will be denoted by $\frac{1}{2}v$. Thus

4·541 $$\varsigma_a \sin \tfrac{1}{2} v = -\varkappa_1 \cos \tfrac{1}{2} v, \quad \varsigma_a \cos \tfrac{1}{2} v = \varkappa_2 \sin \tfrac{1}{2} v,$$

implying

4·542 $$\varkappa_1 \cos^2 \tfrac{1}{2} v + \varkappa_2 \sin^2 \tfrac{1}{2} v = 0,$$

an equation which is of course deducible immediately from Euler's formula 4·444. The direction making an angle $\frac{1}{2}v$ with the first principal direction is one of the directions of the second asymptotic tangent, and is denoted by OK and called the second asymptotic direction. The angle v is an angle from one asymptotic tangent to the other, and is given with as little ambiguity as possible by the equation

4·543 $$B - A \cos v = 0,$$

a corollary of 4·446.

4·55. In the use of the asymptotic directions OJ, OK as directions of reference, there is an embarrassing choice, for the bilinear curvature κ_{JK} and the asymptotic torsion ς_a are connected by the relation

4·551 $$\kappa_{JK} = -\varsigma_a \sin v.$$

For any pair of tangential directions,

4·552 $$\kappa_{ST} \sin^2 v = \kappa_{JK} (\sin \epsilon_{SK} \sin \epsilon_{JT} + \sin \epsilon_{JS} \sin \epsilon_{TK}),$$

and for a single direction

4·553 $$\kappa_n \sin^2 v = 2\kappa_{JK} \sin \epsilon_{TK} \sin \epsilon_{JT},$$

4·554 $$\varsigma_g \sin^2 v = \kappa_{JK} \sin (\epsilon_{TK} - \epsilon_{JT});$$

the last formula can be replaced by

4·555 $$\varsigma_g \sin v = -\varsigma_a \sin (\epsilon_{TK} - \epsilon_{JT}).$$

The principal curvatures are given by

4·556 $$\varkappa_1 = -\varsigma_a \tan \tfrac{1}{2} v, \quad \varkappa_2 = \varsigma_a \cot \tfrac{1}{2} v,$$

and therefore

4·557 $$B = \varsigma_a \cot v, \quad A = \varsigma_a \operatorname{cosec} v, \quad K = -\varsigma_a^2.$$

4·56. From asymptotic tangents are defined asymptotic lines; these on a united anticlastic region without singular or parabolic points compose two families, every point lying on one member of each family. The relation of an asymptotic line to a surface is in a sense more intimate than that of a line of curvature. If an asymptotic line has curvature κ and normal angle ϖ, the normal curvature, which is zero, is $\kappa \cos \varpi$, and three cases are distinguishable: if κ is not zero, then $\cos \varpi$ must be zero; if a point where κ is zero is a limit of points where κ is not zero, continuity requires $\cos \varpi$ to be zero there also; if κ is zero everywhere on the line, the line is straight, and while as a curve in space it has no determinate principal normal at any point, to assign it in its capacity as asymptotic line a definite normal by the *convention** that $\cos \varpi$ is zero leads inevitably to consistent interpretations of general theorems. Thus $\cos \varpi$ is zero at every point of any asymptotic line, and continuous variation of ϖ being on this account out of the question, there is far more gain than loss in a further convention to fix absolutely the value of ϖ, which is taken to be $\frac{1}{2}\pi$:

4·561. *The normal angle of an asymptotic line on a surface is everywhere a right angle.*

In other words,

4·562. *At every point of an asymptotic line on a surface the principal normal to the line is its tangential normal and the binormal is the normal to the surface;*

further, because ϖ is constant,

4·563. *The torsion of an asymptotic line is its geodesic torsion,*

that is, is the asymptotic torsion of the surface in the direction of the line, and because ϖ is a positive right angle,

4·564†. *The curvature of an asymptotic line is its geodesic curvature,*

in sign as well as in amount. In consequence of 4·563, theorems concerning asymptotic torsions may be read narrowly as theorems

* A straight line on a surface is geodesic as well as asymptotic, and *as a geodesic* has for principal normal the normal to the surface.

† This is one of the theorems to whose simplicity the convention that curvature itself must be positive is fatal. The vanishing of $\cos \varpi$ is consistent with a value $-\frac{1}{2}\pi$ for ϖ, and if the direction OP is *predetermined* by the sign of κ, two cases have to be admitted; either ϖ itself, or a symbol for $\sin \varpi$, must then be retained if the cases are to be treated together.

concerning the torsions of asymptotic lines, and in particular 4·522 and 4·526 imply that

4·565. *The torsions at a hyperbolic point O of the two asymptotic lines through O are equal in magnitude and opposite in sign and their product is the specific curvature of the surface at O.*

It is to be remarked that for an asymptotic line to be straight the specific curvature of the surface need not be zero: as an asymptotic line on a given surface containing it, a straight line has a *definite* torsion which is the rate at which the tangent plane, which in general varies from point to point, rotates about the line; on a ruled surface the rotation disappears if the same plane is the tangent plane at every point of the line, and this is precisely the degenerate case in which K is zero along the whole line.

5. The Bilinear Rate of Change of a Function of Position

5·11. The equation

5·111 $$\Phi^1{}_S = 0$$

is of course true only if OS is tangential to the Φ-surface, but the derived equation

5·112 $$\Phi^2{}_{SP} + \Phi^1 (d\mathbf{1}_S/ds_P) = 0$$

involves no restriction on the direction OP, and leads not only to

5·113 $$\Phi^2{}_{ST} + G\kappa_{ST} = 0,$$

the relation used to establish 4·213, but also in virtue of 3·515 and 2·731 to

5·114 $$\Phi^2{}_{SN} + G\tau_S = 0,$$

where τ_S is the spread of the Φ-family along OS, or the negative of the geodesic curvature of the Φ-orthogonal regarded as a curve on a surface to which OS is tangential.

5·21. By means of 5·113 and 5·114 the bilinear rate of change $\Phi^2{}_{PQ}$ can be transformed whenever the direction OP is tangential to the Φ-surface: if the vector $\mathbf{1}_Q$ is the sum $q_T + s_N$ where OT is tangential, then $\Phi^2{}_{SQ}$ is $q\Phi^2{}_{ST} + s\Phi^2{}_{SN}$ and therefore

5·211 $$\Phi^2{}_{SQ} + G(q\kappa_{ST} + s\tau_S) = 0.$$

If OP is the normal direction ON the change is of another kind; from the definition of the slope,

5·212 $$\Phi^1{}_N = G,$$

and therefore along any curve in a direction OQ

5·213 $$d\Phi^1{}_N/ds_Q = G^1{}_Q;$$

but by 1·722 and 3·515

5·214 $$d\Phi^1{}_N/ds_Q = \Phi^2{}_{NQ} + \Phi^1(d1_N/ds_Q) = \Phi^2{}_{NQ} + G\mathscr{G}(d1_N/ds_Q)\,1_N,$$

and since a rate of change of a radial is necessarily at right angles to the radial itself, the last term vanishes and there remains the formula

5·215 $$d\Phi^1{}_N/ds_Q = \Phi^2{}_{NQ},$$

which taken with 5·213 shews that

5·216. *Whatever the direction OQ,*

$$\Phi^2{}_{NQ} = G^1{}_Q.$$

5·22. The general relation of 5·216 is a synthesis of the particular relations

5·221 $$\Phi^2{}_{NN} = G^1{}_N,$$

5·222 $$\Phi^2{}_{NS} = G^1{}_S.$$

The first of these can be written in the form

5·223 $$\Phi^2{}_{NN} = d^2\Phi/dn^2,$$

and suggests a reference to 3·32. The second can be compared with 5·114, and since the bilinear rate of change is symmetrical gives

5·224 $$G^1{}_S + G\tau_S = 0,$$

whence*

5·225 $$\tau_S = -\tfrac{1}{2}d(\log G^2)/ds_S:$$

5·226. *The spread of the Φ-family in any tangential direction is the negative of the rate of change of the logarithmic slope of Φ in that direction.*

5·23. If application is to be made of 5·113, 5·114 and 5·216 to $\Phi^2{}_{PQ}$ when the directions OP, OQ are both oblique, the radials 1_P,

* Allowance must be made for the possibility that G is negative, and for this reason the logarithmic slope is *defined* as $\frac{1}{2}\log G^2$.

$\mathbf{1}_Q$ must both be resolved into normal and tangential components. Assuming

5·231 $$\mathbf{1}_P = p_S + r_N, \quad \mathbf{1}_Q = q_T + s_N,$$

the bilinearity of the function implies

5·232 $$\Phi^2{}_{PQ} = pq\Phi^2{}_{ST} + ps\Phi^2{}_{SN} + qr\Phi^2{}_{TN} + rs\Phi^2{}_{NN},$$

and according to the purpose in view the useful transformation will be

5·233 $$\Phi^2{}_{PQ} - rsG^1{}_N = -G\,(ps\tau_S + qr\tau_T + pq\kappa_{ST})$$

or

5·234 $$\Phi^2{}_{PQ} - psG^1{}_S - qrG^1{}_T - rsG^1{}_N = -pqG\kappa_{ST}.$$

5·31. From 3·515

5·311 $$\Phi^1\,(d\mathbf{1}_S/ds_P) = G\mathscr{G}\,(d\mathbf{1}_S/ds_P)\,\mathbf{1}_N,$$

and since $\mathscr{G}\,\mathbf{1}_S\mathbf{1}_N$ is zero,

5·312 $$\mathscr{G}(d\mathbf{1}_S/ds_P)\,\mathbf{1}_N = -\,\mathscr{G}(d\mathbf{1}_N/ds_P)\,\mathbf{1}_S;$$

hence 5·112 is equivalent to

5·313 $$\Phi^2{}_{SP} = G\mathscr{G}(d\mathbf{1}_N/ds_P)\,\mathbf{1}_S,$$

for an arbitrary direction OP and a tangential direction OS. In contrast to this result, $G\mathscr{G}(d\mathbf{1}_N/ds_P)\,\mathbf{1}_N$ is necessarily zero, but $\Phi^2{}_{NP}$ is zero only in special cases: the tangency of OS is *essential* to the truth of 5·313. Multiplication of 5·313 by a scalar shews that as a linear function of a *tangential* vector $\mathbf{s}$ the bilinear function $\Phi^2{}_P\mathbf{s}$ is obtained by multiplying by G the projected product of $d\mathbf{1}_N/ds_P$ and $\mathbf{s}$. In particular, since a rate of change of the radial $\mathbf{1}_N$ is necessarily tangential,

5·314 $$\Phi^2{}_P\,(d\mathbf{1}_N/ds_Q) = G\mathscr{G}\,(d\mathbf{1}_N/ds_P)\,(d\mathbf{1}_N/\,ds_Q)$$

whatever the directions OP, OQ.

5·32. The function $\Phi^2{}_P\,(d\mathbf{1}_N/ds_Q)$ will reappear at a later stage; 5·314 shews that the function is in fact symmetrical in OP and OQ and indicates the geometrical magnitudes with which it is connected, which depend on the relations of OP and OQ to the Φ-surface. If OS, OT are tangential directions, $d\mathbf{1}_N/ds_S$, $d\mathbf{1}_N/ds_T$ are the corresponding Gaussian velocities; $d\mathbf{1}_N/dn$ is the vector of curvature of the orthogonal trajectory. Since neither κ_{ST} nor τ_T

is defined except for tangential directions, the notation described in 1·55 is applicable and it is possible to write

5·321 $$\mathscr{G}(d\mathbf{1}_N/ds_S)(d\mathbf{1}_N/ds_T) = \kappa_{S*}\kappa_{T*},$$

5·322 $$\mathscr{G}(d\mathbf{1}_N/ds_S)(d\mathbf{1}_N/dn) = \kappa_{S*}\tau_*,$$

5·323 $$\mathscr{G}(d\mathbf{1}_N/dn)^2 = \tau_*^2;$$

the last of these functions is the square of the numerical curvature of the trajectory. To discover analytical expressions in which the same projected products are involved, let $\mathbf{q}^{(P)}$ denote temporarily the vector, dependent upon OP, which is such that for an arbitrary direction of OR the value of $\Phi^2{}_{PR}$ is the projection of $\mathbf{q}^{(P)}$ on OR, and let this vector be resolved into a normal and a tangential component. The projection of $\mathbf{q}^{(P)}$ on ON, which by the definition of $\mathbf{q}^{(P)}$ is $\Phi^2{}_{PN}$, is the projection of the normal component of $\mathbf{q}^{(P)}$ on ON, and therefore the normal component of $\mathbf{q}^{(P)}$ is $\Phi^2{}_{PN}\mathbf{1}_N$. And from 5·313 the projection of the tangential component of $\mathbf{q}^{(P)}$ in *any* tangential direction is the same as the projection of $Gd\mathbf{1}_N/ds_P$ in that direction, whence since $Gd\mathbf{1}_N/ds_P$ is itself tangential the tangential component of $\mathbf{q}^{(P)}$ is nothing but $Gd\mathbf{1}_N/ds_P$. Thus,

5·324 $$\mathbf{q}^{(P)} = G\,(d\mathbf{1}_N/ds_P) + \Phi^2{}_{PN}\mathbf{1}_N.$$

But if OP, OQ are any two directions the projected product $\mathscr{G}\mathbf{q}^{(P)}\mathbf{q}^{(Q)}$ is the bilinear scalar function of OP and OQ denoted by $\Phi^2{}_{P*}\Phi^2{}_{Q*}$, and this is calculable with the greatest ease by means of any frame of reference. Hence from 5·324 and the corresponding formula giving $\mathbf{q}^{(Q)}$

5·325 $$\Phi^2{}_{P*}\Phi^2{}_{Q*} = G^2\mathscr{G}(d\mathbf{1}_N/ds_P)(d\mathbf{1}_N/ds_Q) + \Phi^2{}_{PN}\Phi^2{}_{QN}.$$

The three distinct theorems comprehended in 5·325 can be expressed in a variety of forms; among the results are

5·326 $$\Phi^2{}_{S*}\Phi^2{}_{T*} = G^2(\kappa_{S*}\kappa_{T*} + \tau_S\tau_T),$$

5·327 $$\Phi^2{}_{S*}\Phi^2{}_{N*} = G^2\kappa_{S*}\tau_* + G^1{}_S G^1{}_N,$$

5·328 $$G^2\tau_*^2 = (\Phi^2{}_{N*})^2 - (G^1{}_N)^2.$$

5·41. In 5·226 and 4·213 we have two distinct and independent deductions from the symmetry of the bilinear rate of change of a scalar function of position. It is important to observe that *there can be no deductions independent of these two,* a result implied by 1·432: if OS, OT are distinct tangential directions and ON is

normal, the complete symmetry of $\Phi^2{}_{PQ}$ is deducible from and therefore involves no consequences independent of the set of equalities

5·411 $\Phi^2{}_{ST}=\Phi^2{}_{TS},\quad \Phi^2{}_{NS}=\Phi^2{}_{SN},\quad \Phi^2{}_{NT}=\Phi^2{}_{TN},$

of which the first is equivalent to 4·213, and the second and third express for different directions the single theorem 5·226.

5·42. To suppose that such formulae as 5·113, 5·114 and 5·216 assist in the *calculation* of multilinear rates of change is completely to misvalue these formulae. Whatever the system of coordinates, the multilinear rates of change are among the functions most easily found, and in application to particular surfaces and functions it is rather for the sake of the other magnitudes involved that results of this kind are desirable.

6. The Codazzi Function

6·11. The bilinear curvature κ_{RS} is not a function from which a gradient can be formed, for as a rule if the position of O is changed the directions OR, OS cannot remain unaltered. But there is an elegant function which plays as far as possible the part of a gradient, and it is with this function that the present chapter is concerned.

From the equation

6·111 $$\Phi^2{}_{RS}+G\kappa_{RS}=0$$

it follows that if OR, OS are specified functions of the position of O on a curve with direction OT on a Φ-surface, then

6·112 $$(d\Phi^2{}_{RS}/ds_T)+G^1{}_T\kappa_{RS}+G\,(d\kappa_{RS}/ds_T)=0\,;$$

also by 1·722 and 2·511

6·113
$$\begin{aligned}d\Phi^2{}_{RS}/ds_T&=\Phi^3{}_{RST}+\Phi^2{}_S\,(d1_R/ds_T)+\Phi^2{}_R\,(d1_S/ds_T)\\&=\Phi^3{}_{RST}+\Phi^2{}_S\,(\kappa_{RT}1_N+\sigma_T{}^R1_C)+\Phi^2{}_R\,(\kappa_{ST}1_N+\sigma_T{}^S1_D),\end{aligned}$$

where OC, OD make positive right angles with OR, OS, and therefore

6·114
$$\begin{aligned}d\Phi^2{}_{RS}/ds_T&=\Phi^3{}_{RST}+\Phi^2{}_{NS}\kappa_{RT}+\Phi^2{}_{NR}\kappa_{ST}+\Phi^2{}_{CS}\sigma_T{}^R+\Phi^2{}_{RD}\sigma_T{}^S\\&=\Phi^3{}_{RST}+G^1{}_R\kappa_{ST}+G^1{}_S\kappa_{RT}-G\,(\sigma_T{}^R\kappa_{CS}+\sigma_T{}^S\kappa_{RD}),\end{aligned}$$

on substitution from 5·216 and 4·212. Thus 6·112 gives

6·115 $\Phi^3{}_{RST} + G^1{}_R\kappa_{ST} + G^1{}_S\kappa_{RT} + G^1{}_T\kappa_{RS} + G\lambda_{RST} = 0,$

where

6·116 $\lambda_{RST} = (d\kappa_{RS}/ds_T) - \sigma_T{}^R\kappa_{CS} - \sigma_T{}^S\kappa_{RD}.$

The function λ_{RST} defined by 6·116, which I propose to call the Codazzi function, belongs like the bilinear curvature to the geometry of a single surface, for this definition contains no reference to the function Φ. But 6·115 is of value as shewing at once that the value of λ_{RST} depends only on the three directions OR, OS, OT, not on the variation of OR, OS along any particular curve in the direction OT, and moreover that

6·117. *The Codazzi function is linear in each of the three directions on which it depends.*

As a formula for the calculation of the Codazzi function 6·115 may be modified to

6·118

$$\lambda_{RST} = -(\Phi^1{}_N\Phi^3{}_{RST} - \Phi^2{}_{NR}\Phi^2{}_{ST} - \Phi^2{}_{NS}\Phi^2{}_{RT} - \Phi^2{}_{NT}\Phi^2{}_{RS})/(\Phi^1{}_N)^2.$$

6·12. The Codazzi function λ_{RST} takes special forms if two of the directions on which it depends coincide or are perpendicular. Whatever the angle between OS and OT,

6·121 $\lambda_{SST} = (d\kappa_{SS}/ds_T) - 2\sigma_T{}^S\kappa_{DS}$
$= (d\kappa_{SS}/ds_T) - 2\sigma_T{}^S\varsigma_{SS},$

6·122 $\lambda_{STS} = (d\kappa_{ST}/ds_S) - \sigma_S{}^S\kappa_{DT} - \sigma_S{}^T\kappa_{SE}$
$= (d\kappa_{ST}/ds_S) - 2\sigma_S{}^S\varsigma_{ST} - (d\epsilon_{ST}/ds_S)\,\kappa_{SE},$

6·123 $\lambda_{DST} = (d\kappa_{DS}/ds_T) + \sigma_T{}^D\kappa_{SS} - \sigma_T{}^S\kappa_{DD}$
$= (d\kappa_{DS}/ds_T) - 2\sigma_T{}^S\varsigma_{DS},$

6·124 $\lambda_{DTS} = (d\kappa_{DT}/ds_S) + \sigma_S{}^D\kappa_{ST} - \sigma_S{}^T\kappa_{DE}$
$= (d\kappa_{DT}/ds_S) - 2\sigma_S{}^S\varsigma_{DT} - (d\epsilon_{ST}/ds_S)\kappa_{DE}.$

6·13. More familiar functions are among those of a single direction OT which appear as degenerate forms of the Codazzi function and can be regarded as defined by means of a single curve in the direction OT; that the function depends only on the direction and not on any particular curve is in no case self-evident. In the most elementary notation,

6·131 $\lambda_{TTT} = (d\kappa_n/ds) - 2\kappa_g\varsigma_g,$

6·132 $\lambda_{TET} = (d\varsigma_g/ds) + 2\kappa_g(\kappa_n - B).$

Thus λ_{TTT}, the simplest of cubic functions, is the function associated with the name of Laguerre who first shewed it to depend on direction alone, and λ_{TET} is the cubic function of Darboux. As actually given by 6·116,

6·133 $$\lambda_{EET} = (d\kappa_{EE}/ds) - 2\kappa_g \kappa_{ET'},$$

but on substitution from 4·123, this becomes

6·134 $$\lambda_{EET} = 2\,(dB/ds) - \{(d\kappa_n/ds) - 2\kappa_g \varsigma_g\},$$

that is

6·135 $$\lambda_{EET} = 2B^1{}_T - \lambda_{TTT}.$$

On account of the multilinearity of the Codazzi function, λ_{TTE}, λ_{ETE} bear to the direction OE the relations of λ_{EET}, $-\lambda_{TET}$ to OT; hence

6·136 $$\lambda_{TTE} = 2B^1{}_E - \lambda_{EEE},$$

while λ_{ETE} is the negative of the Darboux function of the direction OE; λ_{EEE} is of course the Laguerre function of this last direction.

6·14. Naturally it is when the three directions involved are all principal or all asymptotic that the Codazzi function is most simply expressed. If $\kappa_{g\mathrm{I}}$, $\kappa_{g\mathfrak{z}}$ are the geodesic curvatures of the lines of curvature,

6·141 $$\sigma_{\mathrm{I}}{}^{\mathrm{I}} = \sigma_{\mathrm{I}}{}^{\mathfrak{z}} = \kappa_{g\mathrm{I}}, \quad \sigma_{\mathfrak{z}}{}^{\mathrm{I}} = \sigma_{\mathfrak{z}}{}^{\mathfrak{z}} = \kappa_{g\mathfrak{z}},$$

and therefore

6·142 $$\lambda_{\mathrm{III}} = d\varkappa_{\mathrm{I}}/ds_{\mathrm{I}}, \quad \lambda_{\mathrm{I}\mathfrak{z}\mathrm{I}} = -2A\kappa_{g\mathrm{I}}, \quad \lambda_{\mathfrak{z}\mathfrak{z}\mathrm{I}} = d\varkappa_{\mathfrak{z}}/ds_{\mathrm{I}},$$

6·143 $$\lambda_{\mathrm{II}\mathfrak{z}} = d\varkappa_{\mathrm{I}}/ds_{\mathfrak{z}}, \quad \lambda_{\mathrm{I}\mathfrak{z}\mathfrak{z}} = -2A\kappa_{g\mathfrak{z}}, \quad \lambda_{\mathfrak{z}\mathfrak{z}\mathfrak{z}} = d\varkappa_{\mathfrak{z}}/ds_{\mathfrak{z}},$$

where A denotes as before $\frac{1}{2}(\varkappa_{\mathfrak{z}} - \varkappa_{\mathrm{I}})$, the amplitude of curvature. The corresponding functions for the asymptotic directions OJ, OK are simplified by the relation

6·144 $$\kappa_{JK} = \varsigma_a \sin v\,;$$

if κ_{gJ}, κ_{gK} are the geodesic curvatures, which are the actual curvatures, of the asymptotic lines,

6·145
$$\lambda_{JJJ} = -2\varsigma_a\kappa_{gJ}, \quad \lambda_{JKJ} = (d\varsigma_a/ds_J)\sin v, \quad \lambda_{KKJ} = 2\varsigma_a\,\{\kappa_{gJ} + (dv/ds_J)\},$$

6·146
$$\lambda_{JJK} = -2\varsigma_a\,\{\kappa_{gK} - (dv/ds_K)\}, \quad \lambda_{JKK} = (d\varsigma_a/ds_K)\sin v, \quad \lambda_{KKK} = 2\varsigma_a\kappa_{gK}.$$

6·21. We are now in a position to appreciate the fundamental property of the Codazzi function, which is apparent from 6·115:

6·211. *The Codazzi function is a symmetrical trilinear function of the tangential directions on which it depends.*

That the function λ_{RST} defined by 6·116 is linear in each of the directions OR, OS, OT can be proved without difficulty from the most elementary considerations*; indeed, it is by its linearity in OT that λ_{RST} first attracts attention in the geometry of a single surface. The symmetry of λ_{RST} in the two directions OR, OS is manifest from the symmetry of κ_{RS} in the same directions, but the discovery that λ_{RST} depends on OT in the same way as on OR and OS is both unexpected and fertile.

6·22. Because λ_{RST} is trilinear,

6·221
$$\begin{aligned}\lambda_{RST}\sin^3\omega &= \lambda_{AAA}\sin\beta_R\sin\beta_S\sin\beta_T + \lambda_{BAA}\sin\alpha_R\sin\beta_S\sin\beta_T\\ &\quad+\lambda_{ABA}\sin\beta_R\sin\alpha_S\sin\beta_T + \lambda_{AAB}\sin\beta_R\sin\beta_S\sin\alpha_T\\ &\quad+\lambda_{ABB}\sin\beta_R\sin\alpha_S\sin\alpha_T + \lambda_{BAB}\sin\alpha_R\sin\beta_S\sin\alpha_T\\ &\quad+\lambda_{BBA}\sin\alpha_R\sin\alpha_S\sin\beta_T + \lambda_{BBB}\sin\alpha_R\sin\alpha_S\sin\alpha_T,\end{aligned}$$

and the *complete* symmetry of the function is implied by the trilinearity if the equalities

6·222
$$\lambda_{BAA} = \lambda_{ABA} = \lambda_{AAB}, \quad \lambda_{ABB} = \lambda_{BAB} = \lambda_{BBA}$$

are known for any one pair of distinct directions. On account of the symmetry of κ_{AB}, there is no distinction between λ_{BAA} and λ_{ABA} or between λ_{ABB} and λ_{BAB}, and therefore the equations necessary to imply 6·211 are two only, namely

6·223
$$\lambda_{AAB} = \lambda_{ABA}, \quad \lambda_{ABB} = \lambda_{BBA},$$

which on reference to 6·121 and 6·122 are readily identified with the equations associated with the name of Codazzi:

6·224. *The Codazzi equations for any pair of families of curves of reference express the symmetry of the Codazzi function for the directions of reference and imply the complete symmetry of this function,*

and it is for this reason that I have proposed to attach Codazzi's name to the function itself.

From 1·43,

6·225. *Any two pairs of Codazzi equations are equivalent,*

and this result adds interest to a comparison of different forms which the equations assume.

* See 8·1 below.

6·23. An important interpretation of the Codazzi equations comes from 6·136, which can now be read as a relation between λ_{TET}, the Darboux function of OT, and λ_{EEE}, the Laguerre function of OE:

6·231. *The sum of the Darboux function of any direction OT and the Laguerre function of the perpendicular direction OE is a linear function, equal to twice the rate of change in the latter direction of the mean curvature of the surface.*

Since the two equations

6·232 $$\lambda_{TTE} = \lambda_{TET}, \quad \lambda_{TEE} = \lambda_{EET}$$

differ only in the direction which is denoted by OT, 6·231 implies them both*, and is equivalent to any pair of Codazzi equations.

Angular differentiation gives relations of another kind between the functions of Laguerre and Darboux. If the variable directions are independent,

6·233 $$da_T \lambda_{RST} = \lambda_{RSE}.$$

Hence because the Codazzi function is symmetrical,

6·234 $$da\lambda_{TTT} = 3\lambda_{TTE},$$

6·235 $$da\lambda_{TTE} = 2\lambda_{TEE} - \lambda_{TTT} = 4B^1{}_T - 3\lambda_{TTT};$$

it is easy to express these results in words.

6·31. The Codazzi equations derived from 6·142, 6·143, and 6·145, 6·146 are

6·311 $$2A\kappa_{g\imath} = -d\varkappa_\imath/ds_{\mathfrak{z}}, \quad 2A\kappa_{g\mathfrak{z}} = -d\varkappa_{\mathfrak{z}}/ds_\imath,$$

and

6·312 $$\left[\begin{aligned} 2\varsigma_a\{\kappa_{gJ} + (dv/ds_J)\} &= (d\varsigma_a/ds_K)\sin v, \\ 2\varsigma_a\{\kappa_{gK} - (dv/ds_K)\} &= -(d\varsigma_a/ds_J)\sin v, \end{aligned}\right.$$

and these are inevitably regarded as formulae for the calculation of the geodesic curvatures which they involve. The same view may be taken of the Codazzi equations in general, for although as a rule each equation involves two geodesic curvatures, the pair of equations

6·313 $$\lambda_{AAB} = \lambda_{ABA}, \quad \lambda_{ABB} = \lambda_{BBA}$$

is linear in the pair of geodesic curvatures $\dot{\kappa}_g$, $\ddot{\kappa}_g$, and has for its discriminant $\varsigma_{AB}{}^2 - \varsigma_{AA}\varsigma_{BB}$, which has been seen in 4·472 to be equal to $A^2 \sin^2 \omega$ and therefore vanishes only at an umbilic.

* Formulae equivalent to 6·232 were discovered in 1911 and announced to the Fifth International Congress of Mathematics (Cambridge, 1912; *Proceedings*, vol. 2, p. 34); I have not hitherto published a proof.

6·32. Since any two pairs of Codazzi equations are equivalent, the geodesic curvatures in any one pair of families of reference curves can be calculated from those in any other pair; this is in accordance with 2·425 and 2·426, but if in illustration we deduce κ_{gJ} from 6·311 we shall see the economy effected by the enlarging of our ideas. Because the swerve $\sigma_T{}^S$ is linear in the direction OT,

6·321 $$\kappa_{gJ}+(dv/ds_J)=\sigma_J{}^K=\sigma_\iota{}^K\cos\tfrac{1}{2}v-\sigma_{\mathfrak{z}}{}^K\sin\tfrac{1}{2}v$$
$$=\{\kappa_{g\iota}+\tfrac{1}{2}(dv/ds_\iota)\}\cos\tfrac{1}{2}v-\{\kappa_{g\mathfrak{z}}-\tfrac{1}{2}(dv/ds_{\mathfrak{z}})\}\sin\tfrac{1}{2}v,$$

and therefore from 6·311

6·322 $$2A\left\{\kappa_{gJ}+\frac{dv}{ds_J}\right\}=\left\{\frac{d\varkappa_{\mathfrak{z}}}{ds_\iota}\sin\tfrac{1}{2}v+A\frac{dv}{ds_\iota}\cos\tfrac{1}{2}v\right\}$$
$$-\left\{\frac{d\varkappa_\iota}{ds_{\mathfrak{z}}}\cos\tfrac{1}{2}v-A\frac{dv}{ds_{\mathfrak{z}}}\sin\tfrac{1}{2}v\right\}$$
$$=\left\{\frac{d(\varsigma_a\cot\frac{1}{2}v)}{ds_\iota}\sin\tfrac{1}{2}v+\tfrac{1}{2}\varsigma_a\frac{dv}{ds_\iota}\operatorname{cosec}\tfrac{1}{2}v\right\}$$
$$+\left\{\frac{d(\varsigma_a\tan\frac{1}{2}v)}{ds_{\mathfrak{z}}}\cos\tfrac{1}{2}v-\tfrac{1}{2}\varsigma_a\frac{dv}{ds_{\mathfrak{z}}}\sec\tfrac{1}{2}v\right\}$$
$$=\frac{d\varsigma_a}{ds_\iota}\cos\tfrac{1}{2}v+\frac{d\varsigma_a}{ds_{\mathfrak{z}}}\sin\tfrac{1}{2}v$$
$$=\frac{d\varsigma_a}{ds_K},$$

as anticipated. It would be rash to assume that *every* useful formula for a geodesic curvature is given by some Codazzi equation; in fact an example can be given to the contrary. Identically,

6·323 $$\frac{d\varsigma_a}{ds_\iota}-2\varsigma_a\frac{dv}{ds_\iota}\operatorname{cosec}v=\frac{d(\varsigma_a\cot^2\frac{1}{2}v)}{ds_\iota}\tan^2\tfrac{1}{2}v,$$

6·324 $$\frac{d\varsigma_a}{ds_{\mathfrak{z}}}+2\varsigma_a\frac{dv}{ds_{\mathfrak{z}}}\operatorname{cosec}v=\frac{d(\varsigma_a\tan^2\frac{1}{2}v)}{ds_{\mathfrak{z}}}\cot^2\tfrac{1}{2}v;$$

utilising the relations

6·325 $$\varsigma_a{}^2\cot^4\tfrac{1}{2}v=\varsigma_a{}^3\cot^3\tfrac{1}{2}v/\varsigma_a\tan\tfrac{1}{2}v=-\varkappa_{\mathfrak{z}}{}^3/\varkappa_\iota,$$

6·326 $$\varsigma_a{}^2\tan^4\tfrac{1}{2}v=\varsigma_a{}^3\tan^3\tfrac{1}{2}v/\varsigma_a\cot\tfrac{1}{2}v=-\varkappa_\iota{}^3/\varkappa_{\mathfrak{z}},$$

it is easy to deduce from 6·312 the expression

6·327 $$\kappa_{gJ}=\{d\log(-\varkappa_{\mathfrak{z}}{}^3/\varkappa_\iota)/ds_\iota\}\sin\tfrac{1}{2}v\cos^2\tfrac{1}{2}v$$
$$+\{d\log(-\varkappa_\iota{}^3/\varkappa_{\mathfrak{z}})/ds_{\mathfrak{z}}\}\cos\tfrac{1}{2}v\sin^2\tfrac{1}{2}v,$$

of which Bonnet has made application, but 6·327 is not as it stands a Codazzi equation.

7. The Trilinear Rate of Change of a Function of Position

7·11. In the last section the trilinear rate of change played only the subsidiary part of introducing to our notice the Codazzi function and establishing its symmetry, and for this purpose the variable directions were restricted to be tangential to the Φ-surface. The next task is to investigate formulae involving the same trilinear rate of change with one or more of the directions normal.

7·12. From the elementary formula

7·121 $$\Phi^2{}_{NP} = G^1{}_P,$$

since this implies for any variable t on which OP may depend

7·122 $$\Phi^2{}_N\,(d1_P/dt) = G^1\,(d1_P/dt),$$

it follows that whatever the direction OQ,

7·123 $$\Phi^3{}_{NPQ} + \Phi^2{}_P\,(d1_N/ds_Q) = G^2{}_{PQ}.$$

Here is a simple proof that the function $\Phi^2{}_P\,(d1_N/ds_Q)$ is symmetrical in the directions OP, OQ, a conclusion reached in 5·3, and substitution from any of the formulae of 5·32 gives a corresponding deduction from 7·123. Thus 5·325 gives

7·124 $$\Phi^1{}_N\,\Phi^3{}_{NPQ} + \Phi^2{}_{P*}\,\Phi^2{}_{Q*} = GG^2{}_{PQ} + G^1{}_P\,G^1{}_Q$$

in which no restriction is implied on OP or OQ, and 5·321, 5·322, 5·323 imply

7·125 $$\Phi^3{}_{NST} + G\kappa_{S*}\kappa_{T*} = G^2{}_{ST},$$

7·126 $$\Phi^3{}_{NNT} + G\kappa_{S*}\,\tau_* = G^2{}_{NT},$$

7·127 $$\Phi^3{}_{NNN} + G\kappa_p{}^2 = G^2{}_{NN},$$

κ_p in 7·127 denoting either value of the curvature of the orthogonal trajectory.

7·13. The function $\Phi^3{}_{NPQ}$ is involved not only in the rate of change $d\Phi^2{}_{NP}/ds_Q$ but also in the rate of change $d\Phi^2{}_{PQ}/dn$, and deductions from the symmetry of the trilinear function are to be expected. If however OQ is normal, nothing is to be anticipated that is not deducible from the symmetry of the *bilinear* function $G^2{}_{NP}$; in fact we have from first principles

7·131 $$d\Phi^2{}_{NP}/dn = \Phi^3{}_{NNP} + \Phi^2{}_N\,(d1_P/dn) + \Phi^2{}_P\,(d1_N/dn),$$

and since

7·132 $$\Phi^2{}_N\,(d1_P/dn) = G^1\,(d1_P/dn),$$

the comparison of 7·131 with 7·123 yields only the identity

7·133 $$dG^1{}_P/dn = G^2{}_{NP} + G^1(d1_P/dn).$$

But $d\Phi^2{}_{ST}/dn$ repays examination.

7·21. Expanding $d\Phi^2{}_{ST}/dn$ in the usual way and substituting from 2·741 we have

7·211 $$\begin{aligned} d\Phi^2{}_{ST}/dn &= \Phi^3{}_{NST} + \Phi^2{}_T(\sigma_N{}^S 1_D + \tau_S 1_N) + \Phi^2{}_S(\sigma_N{}^T 1_E + \tau_T 1_N) \\ &= \Phi^3{}_{NST} + \Phi^2{}_{DT}\sigma_N{}^S + \Phi^2{}_{SE}\sigma_N{}^T + \Phi^2{}_{NT}\tau_S + \Phi^2{}_{NS}\tau_T; \end{aligned}$$

on the other hand, from 4·212,

7·212 $$d\Phi^2{}_{ST}/dn = -d(G\kappa_{ST})/dn = -G(d\kappa_{ST}/dn) - G^1{}_N\kappa_{ST}.$$

Hence from 4·212 and 5·114

7·213

$$\Phi^3{}_{NST} + G^1{}_N\kappa_{ST} + G\{(d\kappa_{ST}/dn) - \sigma_N{}^S\kappa_{DT} - \sigma_N{}^T\kappa_{SE} - 2\tau_S\tau_T\} = 0,$$

or in a form analogous to that of 6·118,

7·214 $$(d\kappa_{ST}/dn) - \sigma_N{}^S\kappa_{DT} - \sigma_N{}^T\kappa_{SE} = -(\Phi^1{}_N\Phi^3{}_{NST} - \Phi^2{}_{NN}\Phi^2{}_{ST} - 2\Phi^2{}_{NS}\Phi^2{}_{NT})/(\Phi^1{}_N)^2.$$

7·22. When 7·213 is compared with 7·125 the function Φ itself disappears, surviving only in the slope:

7·221 $$G^2{}_{ST} + G^1{}_N\kappa_{ST} + G\{(d\kappa_{ST}/dn) - \sigma_N{}^S\kappa_{DT} - \sigma_N{}^T\kappa_{SE} - 2\tau_S\tau_T - \kappa_{S*}\kappa_{T*}\} = 0.$$

An algebraical transformation reduces the number of terms in this equation. Let T denote $1/G$, which is the arc function of the trajectory with respect to the variable Φ, and has been called the spaciousness of the family; then identically, for *any* directions OP, OQ,

7·222 $$T^1{}_P = -G^{-2}G^1{}_P,$$

7·223 $$T^2{}_{PQ} = -G^{-2}G^2{}_{PQ} + 2G^{-3}G^1{}_P G^1{}_Q,$$

so that

7·224 $$GG^2{}_{PQ} - 2G^1{}_P G^1{}_Q = -T^{-3}T^2{}_{PQ}.$$

But for *tangential* directions OS, OT,

7·225 $$G^1{}_S G^1{}_T = G^2\tau_S\tau_T.$$

Hence

7·226 $$(d\kappa_{ST}/dn) - \sigma_N{}^S\kappa_{DT} - \sigma_N{}^T\kappa_{SE} - \kappa_{S*}\kappa_{T*} = (T^2{}_{ST} + T^1{}_N\kappa_{ST})/T,$$

or in another form, involving Φ but separating completely the geometrical from the analytical terms,

$$7{\cdot}227 \quad (d\kappa_{ST}/dn) - \sigma_N{}^S \kappa_{DT} - \sigma_N{}^T \kappa_{SE} - \kappa_{S*}\kappa_{T*} = \Phi^1{}_N T^2{}_{ST} - T^1{}_N \Phi^2{}_{ST}.$$

7·23. Particular cases of 7·214 are

$$7{\cdot}231 \quad (d\kappa_{TT}/dn) - 2\sigma_N{}^T \varsigma_{TT} = -\{\Phi^1{}_N \Phi^3{}_{NTT} - \Phi^2{}_{NN}\Phi^2{}_{TT} - 2(\Phi^2{}_{NT})^2\}/(\Phi^1{}_N)^2,$$

$$7{\cdot}232 \quad (d\kappa_{ET}/dn) - 2\sigma_N{}^T \varsigma_{ET} = -\{\Phi^1{}_N \Phi^3{}_{NET} - \Phi^2{}_{NN}\Phi^2{}_{ET} - 2\Phi^2{}_{NE}\Phi^2{}_{NT}\}/(\Phi^1{}_N)^2,$$

and the corresponding cases of 7·226 are

$$7{\cdot}233 \quad (d\kappa_{TT}/dn) - 2\sigma_N{}^T \varsigma_{TT} - \kappa_{T*}{}^2 = (T^2{}_{TT} + T^1{}_N \kappa_{TT})/T,$$

$$7{\cdot}234 \quad (d\kappa_{ET}/dn) - 2\sigma_N{}^T \varsigma_{ET} - \kappa_{E*}\kappa_{T*} = (T^2{}_{ET} + T^1{}_N \kappa_{ET})/T;$$

to appreciate the last two formulae, we must recognise $\kappa_{T*}{}^2$ as $\kappa_{TT}{}^2 + \kappa_{ET}{}^2$, the square of the amount of the spin of the tangent plane along OT, and $\kappa_{E*}\kappa_{T*}$ as $\kappa_{ET}\kappa_{TT} + \kappa_{EE}\kappa_{TE}$, that is, as $2B\kappa_{ET}$.

7·24. Two symmetrical bilinear functions in a plane OAB are identical if they are equal for the pair of directions OA, OA, for the pair of directions OA, OB, and for the pair of directions OB, OB. Hence any one group of three independent particular cases of 7·214 or 7·226 is equivalent to any other group.

If OT is a principal direction, $\kappa_{T*}{}^2$ is the square of the corresponding principal curvature and ς_{TT} is zero; hence*

7·241. *If OC is a principal direction on a Φ-surface and $\varkappa$ is the corresponding curvature, then*

$$(d\varkappa/dn) - \varkappa^2 = (T^2{}_{CC} + T^1{}_N \varkappa)/T,$$

where T is the reciprocal of the slope of Φ, and the rate of change of $\varkappa$ along the orthogonal trajectory of the Φ-family is given directly by

$$d\varkappa/dn = -\{\Phi^1{}_N \Phi^3{}_{NCC} - \Phi^2{}_{NN}\Phi^2{}_{CC} - 2(\Phi^2{}_{NC})^2\}/(\Phi^1{}_N)^2.$$

* Since the truth of this theorem for one of the principal directions is not deducible from its truth for the other, there are two independent formulae involved in 7·241. These are the formulae whose existence was inferred by Forsyth in 1903 (*Phil. Trans. Roy. Soc. Lond.*, Ser. A, vol. 202, p. 333) from an enumeration of invariants; discovered by the methods described here and translated into a form to require no explanation they were announced to the London Mathematical Society (*Proc. L.M.S.*, vol. 16, p. xxvii) in 1918.

The form taken by 7·232 and 7·234 for a principal direction, that is, by 7·214 and 7·226 when OS, OT are principal directions at right angles, is quite different; $\kappa_{\mathrm{I}\mathfrak{z}}$ being zero on every surface, $d\kappa_{\mathrm{I}\mathfrak{z}}/dn$ is zero, and $\kappa_{\mathrm{I}*}\kappa_{\mathfrak{z}*}$, a multiple of $\kappa_{\mathrm{I}\mathfrak{z}}$, is zero also. Thus only two terms of 7·234 and three of 7·232 survive, and since $\varsigma_{\mathrm{I}\mathfrak{z}}$ is A, the amplitude of curvature, and in the swings $\sigma_N{}^{\mathrm{I}}$ and $\varsigma_N{}^{\mathfrak{z}}$ is to be recognised the magnitude described as the twist of the family,

7·242. *The twist ϖ of the family of surfaces associated with a function Φ is given in terms of Φ itself by*

$$2A\varpi = (\Phi^1{}_N \Phi^3{}_{N\mathrm{I}\mathfrak{z}} - 2\Phi^2{}_{N\mathrm{I}} \Phi^2{}_{N\mathfrak{z}})/(\Phi^1{}_N)^2,$$

and in terms of the reciprocal of the slope of Φ by

$$2A\varpi = -T^2{}_{\mathrm{I}\mathfrak{z}}/T.$$

Of the results of applying 7·214 and 7·226 to the asymptotic directions OJ, OK the most elegant is

7·243 $$(d\varsigma_a/dn)\sin v - \varsigma_a{}^2 \cos v = -T^2{}_{JK}/T,$$

which can easily be verified from 7·241 and 7·242.

7·31. To deduce from 7·242 formulae for evaluating the twist when Φ is given as a function of Cartesian coordinates x, y, z is a simple matter, but requires some preliminary investigations to which the theory of multilinear functions is not essential.

By its definition, the gradient $\mathbf{G}$ of the function Φ is the vector whose projections are Φ_x, Φ_y, Φ_z. The slope G therefore satisfies the equation

7·311 $$G^2 = -\Upsilon^{-2} \begin{vmatrix} 1 & \cos\gamma & \cos\beta & \Phi_x \\ \cos\gamma & 1 & \cos\alpha & \Phi_y \\ \cos\beta & \cos\alpha & 1 & \Phi_z \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix},$$

and since points where Φ_x, Φ_y, Φ_z are simultaneously zero are excluded, G is determined throughout the region under consideration by combining this formula with a choice made at a single point. The direction normal to the Φ-surface is the direction in which the vector $\mathbf{G}$ has the amount G, and therefore

7·312. *The direction cosines of the normal to the Φ-surface are*

$$\Phi_x/G, \quad \Phi_y/G, \quad \Phi_z/G.$$

The ratios of the normal direction are $x_{\mathbf{G}}/G$, $y_{\mathbf{G}}/G$, $z_{\mathbf{G}}/G$, where $x_{\mathbf{G}}$, $y_{\mathbf{G}}$, $z_{\mathbf{G}}$ are the components of $\mathbf{G}$ and are therefore given by formulae of which the first is

7·313 $$x_{\mathbf{G}} = \Upsilon^{-2} \begin{vmatrix} \Phi_x & \cos\gamma & \cos\beta \\ \Phi_y & 1 & \cos\alpha \\ \Phi_z & \cos\alpha & 1 \end{vmatrix}.$$

The direction whose ratios are x_T, y_T, z_T is tangential to the Φ-surface if

7·314 $$\Phi_x x_T + \Phi_y y_T + \Phi_z z_T = 0,$$

and from 0·45 it follows that if the directions OS, OT are tangential and ϵ_{ST} is an angle from the first to the second, then

7·315 $$\Upsilon G \left\| \begin{matrix} x_S & y_S & z_S \\ x_T & y_T & z_T \end{matrix} \right\| = (\Phi_x, \Phi_y, \Phi_z) \sin \epsilon_{ST}.$$

Also if OE makes a positive right angle with OT round ON, then OT makes a positive right angle with ON round OE and therefore by 0·45

7·316 $$G(l_E, m_E, n_E) = \Upsilon \left\| \begin{matrix} x_{\mathbf{G}} & y_{\mathbf{G}} & z_{\mathbf{G}} \\ x_T & y_T & z_T \end{matrix} \right\|,$$

where $x_{\mathbf{G}}$, $y_{\mathbf{G}}$, $z_{\mathbf{G}}$ have the values typified in 7·313, and by 0·46

7·317 $$\Upsilon G(x_E, y_E, z_E) = \left\| \begin{matrix} \Phi_x & \Phi_y & \Phi_z \\ l_T & m_T & n_T \end{matrix} \right\|.$$

7·32. The bilinear curvature of the Φ-surface in the pair of directions OS, OT is shewn by the comparison of 4·212 with 3·311 to be given by

7·321

$$G\kappa_{ST} = -(\Phi_{xx}, \Phi_{yy}, \Phi_{zz}, \Phi_{yz}, \Phi_{zx}, \Phi_{xy} \between x_S, y_S, z_S \between x_T, y_T, z_T),$$

so that in particular for the normal curvature in the direction OT,

7·322 $$G\kappa_n = -(\Phi_{xx}, \Phi_{yy}, \Phi_{zz}, \Phi_{yz}, \Phi_{zx}, \Phi_{xy} \between x_T, y_T, z_T)^2.$$

Combining 7·321 with 7·317 we have

7·323 $$\Upsilon G^2 \kappa_{SE} = \begin{vmatrix} \Phi_{xx} x_S + \Phi_{yx} y_S + \Phi_{zx} z_S & l_T & \Phi_x \\ \Phi_{xy} x_S + \Phi_{yy} y_S + \Phi_{zy} z_S & m_T & \Phi_y \\ \Phi_{xz} x_S + \Phi_{yz} y_S + \Phi_{zz} z_S & n_T & \Phi_z \end{vmatrix};$$

hence in terms of ratios alone

7·324 $2\Upsilon G^2 \varsigma_{ST}$

$$= \begin{vmatrix} \Phi_{xx}x_S + \Phi_{yx}y_S + \Phi_{zx}z_S & x_T + y_T\cos\gamma + z_T\cos\beta & \Phi_x \\ \Phi_{xy}x_S + \Phi_{yy}y_S + \Phi_{zy}z_S & x_T\cos\gamma + y_T + z_T\cos\alpha & \Phi_y \\ \Phi_{xz}x_S + \Phi_{yz}y_S + \Phi_{zz}z_S & x_T\cos\beta + y_T\cos\alpha + z_T & \Phi_z \end{vmatrix} + \begin{vmatrix} \Phi_{xx}x_T + \Phi_{yx}y_T + \Phi_{zx}z_T & x_S + y_S\cos\gamma + z_S\cos\beta & \Phi_x \\ \Phi_{xy}x_T + \Phi_{yy}y_T + \Phi_{zy}z_T & x_S\cos\gamma + y_S + z_S\cos\alpha & \Phi_y \\ \Phi_{xz}x_T + \Phi_{yz}y_T + \Phi_{zz}z_T & x_S\cos\beta + y_S\cos\alpha + z_S & \Phi_z \end{vmatrix},$$

and the geodesic torsion in the direction OT is given by

7·325

$$\Upsilon G^2 \varsigma_g = \begin{vmatrix} \Phi_{xx}x_T + \Phi_{yx}y_T + \Phi_{zx}z_T & x_T + y_T\cos\gamma + z_T\cos\beta & \Phi_x \\ \Phi_{xy}x_T + \Phi_{yy}y_T + \Phi_{zy}z_T & x_T\cos\gamma + y_T + z_T\cos\alpha & \Phi_y \\ \Phi_{xz}x_T + \Phi_{yz}y_T + \Phi_{zz}z_T & x_T\cos\beta + y_T\cos\alpha + z_T & \Phi_z \end{vmatrix}.$$

It is convenient to write

7·326 $$\Upsilon^2 \Xi^{xx} = \begin{vmatrix} \Phi_{xx} & 1 & \Phi_x \\ \Phi_{xy} & \cos\gamma & \Phi_y \\ \Phi_{xz} & \cos\beta & \Phi_z \end{vmatrix},$$

$$2\Upsilon^2 \Xi^{yz} = \begin{vmatrix} \Phi_{yx} & \cos\beta & \Phi_x \\ \Phi_{yy} & \cos\alpha & \Phi_y \\ \Phi_{yz} & 1 & \Phi_z \end{vmatrix} + \begin{vmatrix} \Phi_{zx} & \cos\gamma & \Phi_x \\ \Phi_{zy} & 1 & \Phi_y \\ \Phi_{zz} & \cos\alpha & \Phi_z \end{vmatrix},$$

and so on, and to use Ξ^{zy}, Ξ^{xz}, Ξ^{yx} as equivalent to Ξ^{yz}, Ξ^{zx}, Ξ^{xy}; with this notation, 7·324 becomes

7·327

$$G^2 \varsigma_{ST} = \Upsilon\,(\Xi^{xx}, \Xi^{yy}, \Xi^{zz}, \Xi^{yz}, \Xi^{zx}, \Xi^{xy} \between x_S, y_S, z_S \between x_T, y_T, z_T),$$

and 7·325 takes the form

7·328 $$G^2 \varsigma_g = \Upsilon\,(\Xi^{xx}, \Xi^{yy}, \Xi^{zz}, \Xi^{yz}, \Xi^{zx}, \Xi^{xy} \between x_T, y_T, z_T)^2.$$

7·33. There is no need of the theory of multilinear functions in establishing the theorem that the principal curvatures of the Φ-surface are the roots of the equation

7·331

$$\begin{vmatrix} G\varkappa + \Phi_{xx} & G\varkappa\cos\gamma + \Phi_{yx} & G\varkappa\cos\beta + \Phi_{zx} & \Phi_x \\ G\varkappa\cos\gamma + \Phi_{xy} & G\varkappa + \Phi_{yy} & G\varkappa\cos\alpha + \Phi_{zy} & \Phi_y \\ G\varkappa\cos\beta + \Phi_{xz} & G\varkappa\cos\alpha + \Phi_{yz} & G\varkappa + \Phi_{zz} & \Phi_z \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix} = 0.$$

Applying to 7·321 the determinantal identity

7·332
$$\begin{vmatrix} R_{SS} & R_{TS} \\ R_{ST} & R_{TT} \end{vmatrix}$$
$$= - \begin{vmatrix} R^{xx} & R^{yx} & R^{zx} & y_S z_T - z_S y_T \\ R^{xy} & R^{yy} & R^{zy} & z_S x_T - x_S z_T \\ R^{xz} & R^{yz} & R^{zz} & x_S y_T - y_S x_T \\ y_S z_T - z_S y_T & z_S x_T - x_S z_T & x_S y_T - y_S x_T & 0 \end{vmatrix},$$

where on the left-hand side R_{PQ} denotes

$$\Sigma R^{uv} u_P v_Q, \quad u, v = x, y, z,$$

and the nine coefficients of the form R^{uv} are arbitrary, we have in virtue of 7·315

7·333
$$\Upsilon^2 G^4 K = - \begin{vmatrix} \Phi_{xx} & \Phi_{yx} & \Phi_{zx} & \Phi_x \\ \Phi_{xy} & \Phi_{yy} & \Phi_{zy} & \Phi_y \\ \Phi_{xz} & \Phi_{yz} & \Phi_{zz} & \Phi_z \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix},$$

which is also a corollary of 7·331, and applying the same identity to 7·327, we have similarly

7·334
$$G^6 A^2 = \begin{vmatrix} \Xi^{xx} & \Xi^{yx} & \Xi^{zx} & \Phi_x \\ \Xi^{xy} & \Xi^{yy} & \Xi^{zy} & \Phi_y \\ \Xi^{xz} & \Xi^{yz} & \Xi^{zz} & \Phi_z \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix};$$

in making these deductions we may take an arbitrary pair of tangential directions and appeal to 4·468 and 4·472, or we may take a pair of principal directions and remember that $\kappa_{\text{I}\mathfrak{z}}$, ς_{II}, $\varsigma_{\mathfrak{z}\mathfrak{z}}$ all vanish.

7·34. For the calculation of the twist, or indeed of the value of any symmetrical bilinear function when its arguments are the principal directions of a Φ-surface, it is not necessary to calculate the individual ratios of the principal directions; it is sufficient to discover the values of the six combinations $x_{\text{I}} x_{\mathfrak{z}}$, $y_{\text{I}} y_{\mathfrak{z}}$, $z_{\text{I}} z_{\mathfrak{z}}$, $y_{\text{I}} z_{\mathfrak{z}} + z_{\text{I}} y_{\mathfrak{z}}$, $z_{\text{I}} x_{\mathfrak{z}} + x_{\text{I}} z_{\mathfrak{z}}$, $x_{\text{I}} y_{\mathfrak{z}} + y_{\text{I}} x_{\mathfrak{z}}$, and this we proceed to do.

It is easy to find five linear functions of these six combinations which necessarily vanish: since $\kappa_{\text{I}\mathfrak{z}}$ is zero, 7·321 gives one such, and because the principal directions are perpendicular,

$$x_{\text{I}} x_{\mathfrak{z}} + y_{\text{I}} y_{\mathfrak{z}} + z_{\text{I}} z_{\mathfrak{z}} + (y_{\text{I}} z_{\mathfrak{z}} + z_{\text{I}} y_{\mathfrak{z}}) \cos\alpha + (z_{\text{I}} x_{\mathfrak{z}} + x_{\text{I}} z_{\mathfrak{z}}) \cos\beta$$
$$+ (x_{\text{I}} y_{\mathfrak{z}} + y_{\text{I}} x_{\mathfrak{z}}) \cos\gamma = 0;$$

also multiplying the conditions of tangency

$$\Phi_x x_{\iota} + \Phi_y y_{\iota} + \Phi_z z_{\iota} = 0, \quad \Phi_x x_{\mathfrak{z}} + \Phi_y y_{\mathfrak{z}} + \Phi_z z_{\mathfrak{z}} = 0$$

by $x_{\mathfrak{z}}, x_{\iota}$ and adding we have

$$2\Phi_x x_{\iota} x_{\mathfrak{z}} + \Phi_z (z_{\iota} x_{\mathfrak{z}} + x_{\iota} z_{\mathfrak{z}}) + \Phi_y (x_{\iota} y_{\mathfrak{z}} + y_{\iota} x_{\mathfrak{z}}) = 0,$$

and similarly

$$2\Phi_y y_{\iota} y_{\mathfrak{z}} + \Phi_z (y_{\iota} z_{\mathfrak{z}} + z_{\iota} y_{\mathfrak{z}}) + \Phi_x (x_{\iota} y_{\mathfrak{z}} + y_{\iota} x_{\mathfrak{z}}) = 0,$$

$$2\Phi_z z_{\iota} z_{\mathfrak{z}} + \Phi_y (y_{\iota} z_{\mathfrak{z}} + z_{\iota} y_{\mathfrak{z}}) + \Phi_x (z_{\iota} x_{\mathfrak{z}} + x_{\iota} z_{\mathfrak{z}}) = 0.$$

Hence

7·341. *With any Cartesian frame, the six expressions* $x_{\iota} x_{\mathfrak{z}}$, $y_{\iota} y_{\mathfrak{z}}$, $z_{\iota} z_{\mathfrak{z}}$, $y_{\iota} z_{\mathfrak{z}} + z_{\iota} y_{\mathfrak{z}}$, $z_{\iota} x_{\mathfrak{z}} + x_{\iota} z_{\mathfrak{z}}$, $x_{\iota} y_{\mathfrak{z}} + y_{\iota} x_{\mathfrak{z}}$ *are proportional to the five-rowed determinants of the matrix*

$$\left\| \begin{array}{cccccc} \Phi_{xx} & \Phi_{yy} & \Phi_{zz} & \Phi_{yz} & \Phi_{zx} & \Phi_{xy} \\ 1 & 1 & 1 & \cos\alpha & \cos\beta & \cos\gamma \\ 2\Phi_x & 0 & 0 & 0 & \Phi_z & \Phi_y \\ 0 & 2\Phi_y & 0 & \Phi_z & 0 & \Phi_x \\ 0 & 0 & 2\Phi_z & \Phi_y & \Phi_x & 0 \end{array} \right\|.$$

To find the factor which enables us to replace the proportionality by equality, we remark that 7·341 implies that

7·342. *For arbitrary values of f, g, h, the determinant*

$$\left| \begin{array}{cccccc} f^2 & g^2 & h^2 & gh & hf & fg \\ \Phi_{xx} & \Phi_{yy} & \Phi_{zz} & \Phi_{yz} & \Phi_{zx} & \Phi_{xy} \\ 1 & 1 & 1 & \cos\alpha & \cos\beta & \cos\gamma \\ 2\Phi_x & 0 & 0 & 0 & \Phi_z & \Phi_y \\ 0 & 2\Phi_y & 0 & \Phi_z & 0 & \Phi_x \\ 0 & 0 & 2\Phi_z & \Phi_y & \Phi_x & 0 \end{array} \right|$$

is a multiple of the product $(fx_{\iota} + gy_{\iota} + hz_{\iota})(fx_{\mathfrak{z}} + gy_{\mathfrak{z}} + hz_{\mathfrak{z}})$, and we evaluate this product in another way.

If we write temporarily m, n for $fx_{\iota} + gy_{\iota} + hz_{\iota}$, $fx_{\mathfrak{z}} + gy_{\mathfrak{z}} + hz_{\mathfrak{z}}$, then identically

$$nx_{\iota} - mx_{\mathfrak{z}} = g(x_{\iota} y_{\mathfrak{z}} - y_{\iota} x_{\mathfrak{z}}) - h(z_{\iota} x_{\mathfrak{z}} - x_{\iota} z_{\mathfrak{z}}),$$

and therefore by 7·315

$$\Upsilon G(nx_{\iota} - mx_{\mathfrak{z}}) = g\Phi_z - h\Phi_y;$$

similarly

$$\Upsilon G(ny_{\iota} - my_{\mathfrak{z}}) = h\Phi_x - f\Phi_z, \quad \Upsilon G(nz_{\iota} - mz_{\mathfrak{z}}) = f\Phi_y - g\Phi_x.$$

But because ς_{ST} is symmetrical and bilinear, to replace x_T, y_T, z_T by $nx_\iota - mx_\mathfrak{z}$, $ny_\iota - my_\mathfrak{z}$, $nz_\iota - mz_\mathfrak{z}$ on the right of 7·325 is to replace ς_g by $n^2\varsigma_{\iota\iota} - 2nm\varsigma_{\iota\mathfrak{z}} + m^2\varsigma_{\mathfrak{z}\mathfrak{z}}$ on the left of the same equation; recalling that $\varsigma_{\iota\iota}$, $\varsigma_{\iota\mathfrak{z}}$, $\varsigma_{\mathfrak{z}\mathfrak{z}}$ have the values 0, A, 0 and replacing the product nm by its value we have the equation

7·343 $$-2\Upsilon^3 G^4 A\,(fx_\iota + gy_\iota + hz_\iota)(fx_\mathfrak{z} + gy_\mathfrak{z} + hz_\mathfrak{z})$$
$$= \begin{vmatrix} \begin{array}{l} (g\Phi_z - h\Phi_y)\,\Phi_{xx} \\ \quad + (h\Phi_x - f\Phi_z)\,\Phi_{yx} \\ \qquad + (f\Phi_y - g\Phi_x)\,\Phi_{zx} \end{array} & \begin{array}{l} (g\Phi_z - h\Phi_y) \\ \quad + (h\Phi_x - f\Phi_z)\cos\gamma \\ \qquad + (f\Phi_y - g\Phi_x)\cos\beta \end{array} & \Phi_x \\ \begin{array}{l} (g\Phi_z - h\Phi_y)\,\Phi_{xy} \\ \quad + (h\Phi_x - f\Phi_z)\,\Phi_{yy} \\ \qquad + (f\Phi_y - g\Phi_x)\,\Phi_{zy} \end{array} & \begin{array}{l} (g\Phi_z - h\Phi_y)\cos\gamma \\ \quad + (h\Phi_x - f\Phi_z) \\ \qquad + (f\Phi_y - g\Phi_x)\cos\alpha \end{array} & \Phi_y \\ \begin{array}{l} (g\Phi_z - h\Phi_y)\,\Phi_{xz} \\ \quad + (h\Phi_x - f\Phi_z)\,\Phi_{yz} \\ \qquad + (f\Phi_y - g\Phi_x)\,\Phi_{zz} \end{array} & \begin{array}{l} (g\Phi_z - h\Phi_y)\cos\beta \\ \quad + (h\Phi_x - f\Phi_z)\cos\alpha \\ \qquad + (f\Phi_y - g\Phi_x) \end{array} & \Phi_z \end{vmatrix}.$$

It follows that the determinant in 7·342 is a multiple of the determinant in 7·343, and once attention is drawn to the existence of a connection between them it is a simple matter to reduce the former to the product of the latter by -2. We conclude that

7·344. *The value of the determinant in* 7·342 *is*

$$4\Upsilon^3 G^4 A\,(fx_\iota + gy_\iota + hz_\iota)(fx_\mathfrak{z} + gy_\mathfrak{z} + hz_\mathfrak{z}),$$

and further that

7·345. *The value of the symmetrical bilinear function R_{PQ} of which the expression in terms of Cartesian coordinates is*

$$\Sigma\, R^{uv} u_P v_Q, \quad u, v = x, y, z,$$

when the arguments of the function are the first and second principal directions of the Φ-surface, is given by

$$4\Upsilon^3 G^4 A R_{\iota\mathfrak{z}} = \begin{vmatrix} R^{xx} & R^{yy} & R^{zz} & R^{yz} & R^{zx} & R^{xy} \\ \Phi_{xx} & \Phi_{yy} & \Phi_{zz} & \Phi_{yz} & \Phi_{zx} & \Phi_{xy} \\ 1 & 1 & 1 & \cos\alpha & \cos\beta & \cos\gamma \\ 2\Phi_x & 0 & 0 & 0 & \Phi_z & \Phi_y \\ 0 & 2\Phi_y & 0 & \Phi_z & 0 & \Phi_x \\ 0 & 0 & 2\Phi_z & \Phi_y & \Phi_x & 0 \end{vmatrix}.$$

Determinants of the form occurring here were first used by Darboux, who discovered them in his researches on triply-orthogonal systems, and we shall call them Darboux determinants. We may express 7·345 briefly by saying that *the value of the Darboux determinant which has R^{uv} for the typical element of its first row is* $4\Upsilon^3 G^4 A R_{\iota\zeta}$.

In passing we may mention another expression involving the product $(fx_\iota + gy_\iota + hz_\iota)(fx_\zeta + gy_\zeta + hz_\zeta)$, interesting in itself but as ill adapted as that in 7·343 to giving the value of a bilinear function that is not a product. From the identity

7·346

$$\begin{vmatrix} R_{SS} & R_{TS} & f'x_S + g'y_S + h'z_S \\ R_{ST} & R_{TT} & f'x_T + g'y_T + h'z_T \\ f''x_S + g''y_S + h''z_S & f''x_T + g''y_T + h''z_T & 0 \end{vmatrix}$$
$$= \begin{vmatrix} R^{xx} & R^{yx} & R^{zx} & y_S z_T - z_S y_T & f' \\ R^{xy} & R^{yy} & R^{zy} & z_S x_T - x_S z_T & g' \\ R^{xz} & R^{yz} & R^{zz} & x_S y_T - y_S x_T & h' \\ y_S z_T - z_S y_T & z_S x_T - x_S z_T & x_S y_T - y_S x_T & 0 & 0 \\ f'' & g'' & h'' & 0 & 0 \end{vmatrix},$$

since $$\Xi_{\iota\iota} = 0, \quad \Upsilon \Xi_{\iota\zeta} = G^2 A, \quad \Xi_{\zeta\zeta} = 0,$$

we have

7·347 $$2G^4 A\,(fx_\iota + gy_\iota + hz_\iota)(fx_\zeta + gy_\zeta + hz_\zeta)$$
$$= -\Upsilon^3 \begin{vmatrix} \Xi^{xx} & \Xi^{yx} & \Xi^{zx} & \Phi_x & f \\ \Xi^{xy} & \Xi^{yy} & \Xi^{zy} & \Phi_y & g \\ \Xi^{xz} & \Xi^{yz} & \Xi^{zz} & \Phi_z & h \\ \Phi_x & \Phi_y & \Phi_z & 0 & 0 \\ f & g & h & 0 & 0 \end{vmatrix}.$$

7·35. The application of 7·345 to 7·242 is immediate:

7·351. *At a point which is not an umbilic of the Φ-surface through it, the twist ϖ of the Φ-family is connected with the derivatives of Φ by the formula**

* The formula was first given, in terms of curvilinear coordinates and without proof, to the Fifth International Congress of Mathematicians (Cambridge, 1912; see *Proceedings*, vol. 2, p. 31). Results equivalent to 7·351 and 7·352 in terms of rectangular Cartesian coordinates were proved subsequently by Herman (*Quarterly Journal of Mathematics*, vol. 46, pp. 284 *et seq.*). Algebraical transformations of the determinant involved are to be found in Darboux's treatise on triply-orthogonal systems.

$$8\Upsilon^3 A^2 \varpi = -T^3 \begin{vmatrix} T_{xx} & T_{yy} & T_{zz} & T_{yz} & T_{zx} & T_{xy} \\ \Phi_{xx} & \Phi_{yy} & \Phi_{zz} & \Phi_{yz} & \Phi_{zx} & \Phi_{xy} \\ 1 & 1 & 1 & \cos\alpha & \cos\beta & \cos\gamma \\ 2\Phi_x & 0 & 0 & 0 & \Phi_z & \Phi_y \\ 0 & 2\Phi_y & 0 & \Phi_z & 0 & \Phi_x \\ 0 & 0 & 2\Phi_z & \Phi_y & \Phi_x & 0 \end{vmatrix},$$

where

$$\Upsilon^2/T^2 = -\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \Phi_x \\ \cos\gamma & 1 & \cos\alpha & \Phi_y \\ \cos\beta & \cos\alpha & 1 & \Phi_z \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix},$$

and A is the amplitude of curvature of the Φ-surface.

The value of A^2 in terms of derivatives of Φ is given in 7·334 above.

The alternative expression for the twist given by 7·242 is more complicated, but gives the result explicitly in terms of third derivatives of Φ. If x_G, y_G, z_G have the meanings assigned in 7·31, then $G\Phi^2{}_{NP}$ is a linear function of OP in which the coefficient of x_P is

$$\Phi_{xx}x_G + \Phi_{yx}y_G + \Phi_{zx}z_G,$$

and $G\Phi^3{}_{NPQ}$ is a bilinear function of OP and OQ in which the coefficient of $x_P x_Q$ is

$$\Phi_{xxx}x_G + \Phi_{yxx}y_G + \Phi_{zxx}z_G:$$

7·352. *If the typical element in the first row of a Darboux determinant is*

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \Phi_x \\ \cos\gamma & 1 & \cos\alpha & \Phi_y \\ \cos\beta & \cos\alpha & 1 & \Phi_z \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix} \begin{vmatrix} 1 & \cos\gamma & \cos\beta & \Phi_{xuv} \\ \cos\gamma & 1 & \cos\alpha & \Phi_{yuv} \\ \cos\beta & \cos\alpha & 1 & \Phi_{zuv} \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix}$$

$$-2\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \Phi_{xu} \\ \cos\gamma & 1 & \cos\alpha & \Phi_{yu} \\ \cos\beta & \cos\alpha & 1 & \Phi_{zu} \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix} \begin{vmatrix} 1 & \cos\gamma & \cos\beta & \Phi_{xv} \\ \cos\gamma & 1 & \cos\alpha & \Phi_{yv} \\ \cos\beta & \cos\alpha & 1 & \Phi_{zv} \\ \Phi_x & \Phi_y & \Phi_z & 0 \end{vmatrix}$$

the value of the determinant is $8\Upsilon^7 G^8 A^2 \varpi$.

It is to be remarked that the expression given in this enunciation is *not* in general simply the product of the second derivative T_{uv} by $-\Upsilon^4 G^5$; the difference between the two is a function which is such that its use as a typical element of the first row produces a Darboux determinant that vanishes.

8. Functions of Direction on a Surface

8·11. The fundamental difficulty in applying the theory of multilinear functions to problems connected with any single surface other than a plane is due to the absence of genuine gradients. The directions forming the arguments of such functions as the bilinear curvature and the Codazzi function are essentially tangential, and if the current point varies these tangential directions are necessarily affected.

Suppose the surface to be referred to curvilinear coordinates u, v and let a standard tangential direction OW be associated with each position of O. Then a function $F(Q, R, \ldots)$ of the tangential directions OQ, OR, ... may be described explicitly as a function $F(u, v, \epsilon_{WQ}, \epsilon_{WR}, \ldots)$, and if the directions OQ, OR, ... vary in a given manner with the position of O on a curve in the direction OT, the rate of change of $F(Q, R, \ldots)$ along the curve is given by

8·111 $$\frac{dF}{ds_T} = \frac{\partial F}{\partial u}\frac{du}{ds_T} + \frac{\partial F}{\partial v}\frac{dv}{ds_T} + da_Q F\frac{d\epsilon_{WQ}}{ds_T} + da_R F\frac{d\epsilon_{WR}}{ds_T} + \ldots,$$

that is, by

8·112 $$\frac{dF}{ds_T} = \left\{\frac{\partial F}{\partial u}\frac{du}{ds_T} + \frac{\partial F}{\partial v}\frac{dv}{ds_T} - (da_Q F + da_R F + \ldots)\,\sigma_T{}^W\right\} + (\sigma_T{}^Q da_Q F + \sigma_T{}^R da_R F + \ldots).$$

Since u, v are merely particular functions of position on the surface, the rates of change du/ds_T, dv/ds_T are the linear functions $u^1{}_T$, $v^1{}_T$ of OT, and since OW is assumed to depend only on the position of O, the swerve $\sigma_T{}^W$ also is a linear function of OT. On the other hand, neither the rates of change dF/ds_T, $da_Q F$, $da_R F$, ... nor the swerves $\sigma_T{}^Q$, $\sigma_T{}^R$, ... depend in any way on the actual choice of coordinates u, v and initial direction OW. Thus

8·113. *Associated with any function of direction $F(Q, R, \ldots)$ on a surface there is a function $gdb_T F$ which is linear in the direction*

OT and is such that the rate of change of F along any curve in the direction OT is

$$gdb_T F + \sigma_T{}^Q da_Q F + \sigma_T{}^R da_R F + \ldots.$$

The function $gdb_T F$ will be called the Darboux gradient of F. From 8·112,

8·114. *In terms of coordinates u, v and an initial direction OW, the value of the Darboux gradient $gdb_T F$ is given by*

$$gdb_T F = (\partial F/\partial u)\, u^1{}_T + (\partial F/\partial v)\, v^1{}_T - (da_Q F + da_R F + \ldots)\, \sigma_T{}^W.$$

8·12. For a multilinear function P_{QRS}, the angular derivatives $da_Q P$, $da_R P$, ... have the values P_{BRS}, P_{QCS}, ... where OB, OC, ... make positive right angles with OQ, OR, ...; hence

8·121 $\quad dP_{QRS}/ds_T = gdb_T P_{QRS} + P_{BRS}\,\sigma_T{}^Q + P_{QCS}\,\sigma_T{}^R + \ldots.$

There is another route, open only in the case of multilinear functions, which leads to a similar formula and therefore shews a different aspect of the Darboux gradient. If the function P_{FGH} was defined for *all* sets of directions, there would be a gradient $P_{FGH\,K}$ also defined for all sets of directions, and for tangential directions OQ, OR, OS, ... OT we should have

8·122 $$dP_{QRS}/ds_T = P_{QRS\,T} + P_{BRS}\,\sigma_T{}^Q + P_{QCS}\,\sigma_T{}^R + \ldots$$
$$+ P_{NRS}\,\kappa_{QT} + P_{QNS}\,\kappa_{RT} + \ldots.$$

If the functions P_{NRS}, P_{QNS}, ... which might, of course, be different functions, were known, the function $P_{QRS\,T}$ could be *determined* for tangential arguments by this formula; if P_{QRS} was given in the first place for tangential arguments only, the functions P_{NRS}, P_{QNS}, ... could be assigned arbitrarily, and by comparing 8·122 with 8·121 we see that the Darboux gradient is the gradient found by supposing the functions P_{NRS}, P_{QNS}, ... all to be identically zero.

Nevertheless, the Darboux gradient is a disappointing function. The Codazzi function is the Darboux gradient of the bilinear curvature, and if Φ is a function of position on the surface and $\Phi^1{}_T$ is the linear function $d\Phi/ds_T$, the Darboux gradient of $\Phi^1{}_T$ is the function $(d\Phi^1{}_S/ds_T) - \Phi^1{}_D\,\sigma_T{}^S$, which is in fact symmetrical and is valuable on account of its symmetry. But the Darboux gradients of the Codazzi function and of the function $(d\Phi^1{}_S/ds_T) - \Phi^1{}_D\,\sigma_T{}^S$ prove both to be unsymmetrical; gradients with the symmetry

that is desirable are not yielded by any simple general method, and all that is possible is to discover special devices effective in particular cases.

8·21. To define multilinear rates of change of a regular scalar function of position Φ on the surface, we extend the function to the whole of space in the neighbourhood of the surface by associating with every point on the normal at O the value of Φ at O itself. If two or more normals meet at a point Q, the function so defined may be many-valued at Q; if however the surface has only ordinary points there is a region of space within which no two normals intersect, and within this region Φ is not only single-valued but regular.

To assign the values of Φ outside the surface in the way suggested seems at first no less arbitrary a proceeding than to construct the successive Darboux gradients by defining the functions $\Phi^1{}_N$, $\Phi^2{}_{NT}$, $\Phi^3{}_{NST}, \ldots$ to be zero. But a number of considerations combine to modify this impression: the multilinear rates of change formed by extending the function in any regular way are *necessarily* symmetrical; throughout the whole of differential geometry the straight line is much more than merely the simplest of curves; and the hypothesis made is in fact equivalent only to the assumption that the functions $\Phi^1{}_N$, $\Phi^2{}_{NN}$, $\Phi^3{}_{NNN}, \ldots$, functions in which no arbitrary directions are involved, are all zero.

8·22. If attention is concentrated upon the distribution of Φ on the surface, $\Phi^1{}_N$, $\Phi^2{}_{NN}$, $\Phi^3{}_{NNN}, \ldots$ figure as functions of position only, $\Phi^1{}_T$, $\Phi^2{}_{NT}$, $\Phi^3{}_{NNT}, \ldots$ as linear functions of the one variable direction OT, $\Phi^2{}_{ST}$, $\Phi^3{}_{NST}$, $\Phi^4{}_{NNST}, \ldots$ as bilinear functions of the pair of variable directions OS, OT, and so on. The rate of change of any one of these functions along a curve on the surface is expressible by means of other functions in the set, the typical relation being

$$\mathbf{8{\cdot}221}\quad d\Phi^{h+k}_{N\ NNPQ.\,RS}\Big/ds_T$$
$$= \Phi^{h+k+1}_{N\ NNPQ\ RST} - h\kappa_{T*}\Phi^{h+k}_{N\,.\,N*PQ\ RS}$$
$$+ \sigma_T{}^P\Phi^{h+k}_{N\,..NNAQ...RS} + \sigma_T{}^Q\Phi^{h+k}_{N\ NNPB\,.\,RS} + \ldots$$
$$+ \sigma_T{}^S\Phi^{h+k}_{N\ NNPQ\ .RD}$$
$$+ \kappa_{PT}\Phi^{h+k}_{N\ NNNQ...RS} + \kappa_{QT}\Phi^{h+k}_{N\ .NNNP\ RS} + \ldots$$
$$+ \kappa_{ST}\Phi^{h+k}_{N\ NNNPQ...R},$$

where $OP, OQ, \ldots OR, OS, OT$ are $k+1$ tangential directions and $OA, OB, \ldots OD$ make positive right angles with $OP, OQ, \ldots OS$. Comparison of 8·221 with 8·121 shews the relation of $\Phi^{h+k+1}_{N\ NNPQ\ \ RST}$ to the Darboux gradient of $\Phi^{h+k}_{N..\ NNPQ\ \ RS}$:

8·222 $\Phi^{h+k+1}_{N...NNPQ\ \ RST}$

$$= gdb_T \Phi^{h+k}_{N\ \ NNPQ\ \ RS} + h\kappa_{T*}\Phi^{h+k}_{N\ \ N*PQ\ \ RS}$$
$$- \kappa_{PT}\Phi^{h+k}_{N\ \ NNNQ..RS} - \kappa_{QT}\Phi^{h+k}_{N\ \ NNNP\ \ RS} - \ldots - \kappa_{ST}\Phi^{h+k}_{N\ \ NNNPQ..R};$$

none of the functions $\Phi^2{}_{NT}$, $\Phi^3{}_{NNT}$, $\Phi^4{}_{NNNT}$, ... vanish identically, and therefore unless the surface is a plane, h must be zero for $h\kappa_{T*}\Phi^{h+k}_{N\ \ N*PQ\ \ RS}$ to vanish, and k must be unity for the remaining terms in the difference between the two functions to vanish. That is to say, $\Phi^2{}_{ST}$ is the Darboux gradient of $\Phi^1{}_T$, but there is no similar relation between others of the multilinear functions with which we are dealing unless either the surface is plane or the function Φ has some special relation to the surface.

8·23. It is easy, accepting the assumptions

8·231 $$\Phi^1{}_N = 0, \quad \Phi^2{}_{NN} = 0, \quad \Phi^3{}_{NNN} = 0, \ldots,$$

with the implications

8·232 $$d\Phi^1{}_N/ds_T = 0, \quad d\Phi^2{}_{NN}/ds_T = 0, \quad d\Phi^3{}_{NNN}/ds_T = 0, \ldots,$$

to arrange the formulae included under 8·221 in such an order that each of the multilinear functions is introduced without further reference to space outside the surface than is implied in the occurrence of bilinear curvatures as factors. The first equation is

8·233 $$\Phi^1{}_T = d\Phi/ds_T,$$

from which $\Phi^1{}_T$ may be calculated from a curve lying wholly in the surface. Then since

8·234 $$d\Phi^1{}_N/ds_T = \Phi^2{}_{NT} - \kappa_{T*}\Phi^1{}_*$$

identically, and the rate of change is zero,

8·235 $$\Phi^2{}_{NT} - \kappa_{T*}\Phi^1{}_* = 0;$$

also because $\Phi^1{}_N$ is zero,

8·236 $$\Phi^2{}_{ST} + \sigma_T{}^S\Phi^1{}_D = d\Phi^1{}_S/ds_T.$$

Next come

8·237 $$\Phi^3{}_{NNT} - 2\kappa_{T*}\Phi^2{}_{N*} = 0,$$

8·238 $$\Phi^3{}_{NST} - \kappa_{T*}\Phi^2{}_{S*} + \sigma_T{}^S\Phi^2{}_{ND} = d\Phi^2{}_{NS}/ds_T,$$

8·239
$$\Phi^3{}_{RST} + \kappa_{RT}\Phi^2{}_{NS} + \kappa_{ST}\Phi^2{}_{NR} + \sigma_T{}^R\Phi^2{}_{CS} + \sigma_T{}^S\Phi^2{}_{RD} = d\Phi^2{}_{RS}/ds_T,$$

in which the only fresh functions are $\Phi^3{}_{NNT}$, $\Phi^3{}_{NST}$, $\Phi^3{}_{RST}$, and the process can be continued to any desired extent.

8·24. Emphatically the formulae of the last paragraph and their successors are neither *definitions* of the multilinear functions nor aids to their calculation. For the former part they are unsuitable because neither the symmetry nor the multilinearity of the functions is in evidence in the formulae, for the latter because the rates of change and the swerves contain parts that are not multilinear which it is superfluous to evaluate. To discuss the expression of these multilinear functions by means of curvilinear coordinates on the surface requires an analytical foundation which is beyond the range of this pamphlet, and we must content ourselves with the observation that rather than calculate the functions directly from 8·221 we should combine 8·222 with 8·114 and use the formula,

8·241 $$\Phi^{h+k+1}_{N\ NNPQ\ \ RST}$$
$$= \left(\partial\Phi^{h+k}_{N\ .NNPQ\ \ RS}\big/\partial u\right) u^1{}_T + \left(\partial\Phi^{h+k}_{N\ .NNPQ\ \ RS}\big/\partial v\right) v^1{}_T$$
$$- \sigma_T{}^W\left(\Phi^{h+k}_{N\ NNAQ\ \ RS} + \Phi^{h+k}_{N\ NNPB..RS} + \dots + \Phi^{h+k}_{N\ NNPQ\ \ RD}\right)$$
$$+ h\kappa_{T*}\Phi^{h+k}_{N\ N*PQ\ \ RS}$$
$$- \kappa_{PT}\Phi^{h+k}_{N\ NNNQ\ \ RS} - \kappa_{QT}\Phi^{h+k}_{N\ NNNP..RS} - \dots - \kappa_{ST}\Phi^{h+k}_{N\ NNNPQ\ \ R};$$

but this is not the method actually to be recommended.

The very lack of symmetry which renders the formulae covered by 8·221 unfit to serve as definitions implies that significant relations which do not themselves involve multilinear rates of change are deducible from these formulae. To work out details is interesting—it will be found for example that 8·235 and 8·238 together imply the symmetry of the Codazzi function—but here we will

confine our attention to the simplest problem of the kind, and examine only 8·236.

8·31. The bilinear function $\Phi^2{}_{ST}$ being symmetrical, we have from 8·236,

8·311 $$(d\Phi^1{}_S/ds_T) - \sigma_T{}^S\Phi^1{}_D = (d\Phi^1{}_T/ds_S) - \sigma_S{}^T\Phi^1{}_E;$$

involved in 8·311 are really two families of curves and their orthogonal trajectories, and the equality may be written in the form

8·312 $$\frac{d^2\Phi}{ds_T ds_S} - \sigma_T{}^S\frac{d\Phi}{ds_D} = \frac{d^2\Phi}{ds_S ds_T} - \sigma_S{}^T\frac{d\Phi}{ds_E},$$

or in a different notation as

8·313 $$\frac{d^2\Phi}{d\ddot{s}\,d\dot{s}} - \left(\ddot{\kappa}_g - \frac{d\omega}{ds}\right)\frac{d\Phi}{d\dot{m}} = \frac{d^2\Phi}{d\dot{s}\,d\dot{s}} - \left(\dot{\kappa}_g + \frac{d\omega}{d\dot{s}}\right)\frac{d\Phi}{d\ddot{m}},$$

where $d/d\dot{m}$, $d/d\ddot{m}$ indicate rates of change along the orthogonal trajectories of the families. If the families of curves are everywhere orthogonal, 8·313 becomes

8·314 $$\frac{d^2\Phi}{dm\,ds} - \kappa_{gm}\frac{d\Phi}{dm} = \frac{d^2\Phi}{ds\,dm} + \kappa_{gs}\frac{d\Phi}{ds},$$

where κ_{gs}, κ_{gm} denote the geodesic curvatures of a typical member of a family and of an orthogonal trajectory of the same family.

8·32. We must not fail to observe that if what is being discussed is the variation on a particular surface of a function *already* defined throughout space, the formulae of 8·23 and the transformations of 8·31 are not usually valid. For example, in general when the function and the surface are defined independently of each other,

8·321 $$d\Phi^1{}_S/ds_T = \Phi^2{}_{ST} + \sigma_T{}^S\Phi^1{}_D + \kappa_{ST}\Phi^1{}_N,$$

and for the last term to disappear either $\Phi^1{}_N$ must be zero or OS, OT must be conjugate directions. Of the cases in which the latter condition is satisfied the most important is that in which the two principal directions occur: without any hypothesis as to a relation between Φ and the surface,

8·322 $$\Phi^2{}_{\mathfrak{t}\mathfrak{z}} = \frac{d^2\Phi}{ds_{\mathfrak{z}}ds_{\mathfrak{t}}} - \kappa_{g\mathfrak{z}}\frac{d\Phi}{ds_{\mathfrak{z}}} = \frac{d^2\Phi}{ds_{\mathfrak{t}}ds_{\mathfrak{z}}} + \kappa_{g\mathfrak{t}}\frac{d\Phi}{ds_{\mathfrak{t}}},$$

and so in particular the function $T_{\mathfrak{t}\mathfrak{z}}$ which was shewn in 7·242 to be connected with the twist of the Φ-family is expressible

as $(d^2T/ds_{\mathfrak{z}}ds_{\mathfrak{i}}) - \kappa_{g\mathfrak{z}}(dT/ds_{\mathfrak{z}})$ or as $(d^2T/ds_{\mathfrak{i}}ds_{\mathfrak{z}}) + \kappa_{g\mathfrak{i}}(dT/ds_{\mathfrak{i}})$ although in general $T^1{}_N$ is not zero and $T^2{}_{ST}$ differs from $(d^2T/ds_Tds_S) - \sigma_T{}^S(dT/ds_D)$ by $\kappa_{ST}(dT/dn)$.

8·33. The cases of 8·311 which are most easily appreciated are found in the application to two special families of curves on the surface, the Φ-curves and the Φ-orthogonals. The gradient of Φ is a tangential vector G which is at right angles to the Φ-curve. Points where the gradient is the zero vector being excluded, the amounts of the gradient are separate single-valued functions of position on the surface, and one of these is chosen to be called the slope of Φ; the slope will be denoted by G. The direction in which the gradient has the slope G is defined to be the standard direction of the Φ-orthogonal and of the tangential normal to the Φ-curve, and will be denoted by OM. The direction OL with which OM makes a positive right angle is the standard direction of the Φ-curve.

By definition

8·331 $$\Phi^1{}_L = 0, \quad \Phi^1{}_M = G,$$

whence

8·332 $$\Phi^2{}_{MT} = dG/ds_T = G^1{}_T,$$

8·333 $$\Phi^2{}_{LT} = -G\sigma_T{}^L = -G\{\kappa_{gT} + (d\epsilon_{TL}/ds_T)\}.$$

Thus

8·334. *The geodesic curvature of the Φ-curve is $-\Phi^2{}_{LL}/\Phi^1{}_M$,*

and

8·335. *The geodesic curvature of the Φ-orthogonal can be expressed both as $-\Phi^2{}_{LM}/\Phi^1{}_M$ and as $-\frac{1}{2}d(\log G^2)/ds_L$.*

It may be added that 8·335 is deducible from 2·831 and 5·224, for with the convention by which Φ is extended into space a Φ-surface is the ruled surface composed of the normals to the original surface along a Φ-curve.

8·41. I have not succeeded in continuing satisfactorily the sequence of geometrical functions of which the first two members are the bilinear curvature and the Codazzi function. Differentiation of 6·115 gives

8·411 $$\begin{aligned}\Phi^4{}_{QRST} &+ G^2{}_{QT}\kappa_{RS} + G^2{}_{RT}\kappa_{QS} + G^2{}_{ST}\kappa_{QR} + G^2{}_{RS}\kappa_{QT} \\ &+ G^2{}_{QS}\kappa_{RT} + G^2{}_{QR}\kappa_{ST} + G^1{}_Q\lambda_{RST} + G^1{}_R\lambda_{QST} + G^1{}_S\lambda_{QRT} \\ &+ G^1{}_T\lambda_{QRS} + G^1{}_N(\kappa_{QT}\kappa_{RS} + \kappa_{RT}\kappa_{QS} + \kappa_{ST}\kappa_{QR}) + G\mu_{QRST} = 0,\end{aligned}$$

where

8·412 $\mu_{QRST} = gdb_T\lambda_{QRS} - \kappa_{QT}\kappa_{R*}\kappa_{S*} - \kappa_{RT}\kappa_{Q*}\kappa_{S*} - \kappa_{ST}\kappa_{Q*}\kappa_{R*}$,

and while 8·412 shews that μ_{QRST} depends only on the form of the Φ-surface, 8·411 shews that the function is a symmetrical quadrilinear function of tangential directions. Thus there is no difficulty in the construction of the third member of the sequence of functions, and none is to be anticipated in repeating the process again and again, but no general rule is apparent under which the constructions fall, and it is evident that the formulae rapidly become too complicated to be intelligible without some clue to their composition.

INDEX OF DEFINITIONS

INDEX OF SYMBOLS